Free-edge effects at circular holes in composite laminates under hygrothermomechanical load

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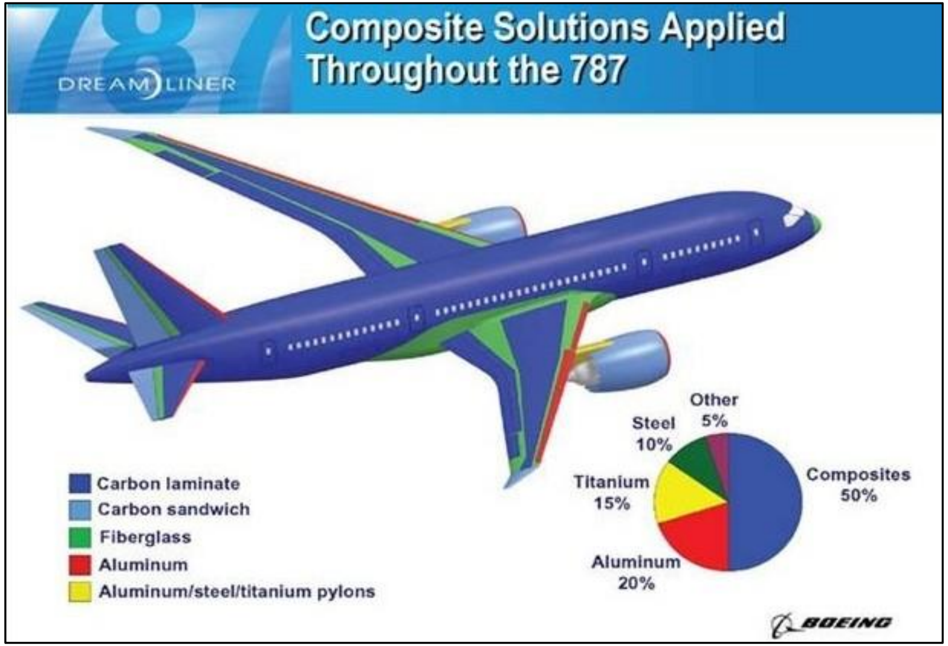
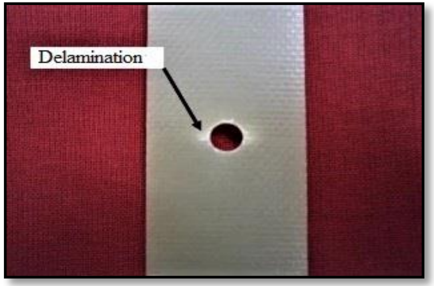
Application and importance





Objective & Contents

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Presenting a complete analytical method to predict the behaviour of 3D stresses sorrounding a hole in composite laminates under mechanical and thermal loads

- Literature
- Finite element (FE) model
- Analytical solution
    - Basic formulations
    - Boundary and continuity conditions
    - Stress solution for the interior region under temperature change
    - Stress solution for the boundary layer region
    - Solution using energy minimization
- Results
- Conclusion and future work

Literature

Kharghani, N., Mittelstedt, C.
Reduction of free-edge effects around a hole of a composite plate using a numerical layup optimization
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Ko, C-C., Lin, C-C.
Method for calculating the interlaminar stresses in symmetric laminates containing a circular hole
AIAA J. **1992**

Lekhnitskii, S.G.
Theory of elasticity of anisotropic body
Holden Day, San Francisco **1963**

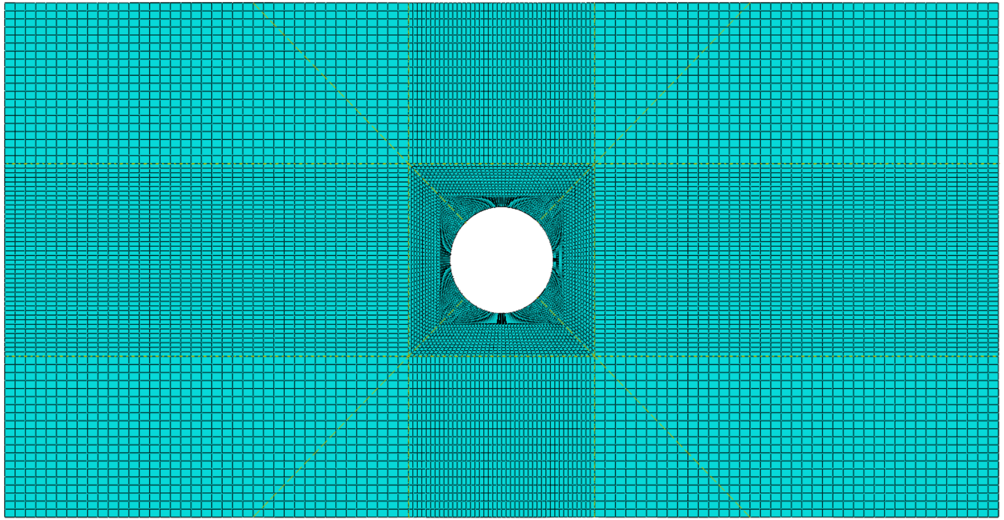
Chaleshtari, M.H.B., Khoramishad, H.
Investigation of the effect of cutout shape on thermal stresses in perforated multilayer composites subjected to heat flux using an analytical method
European J. of Mechanics - A/Solids **2022**

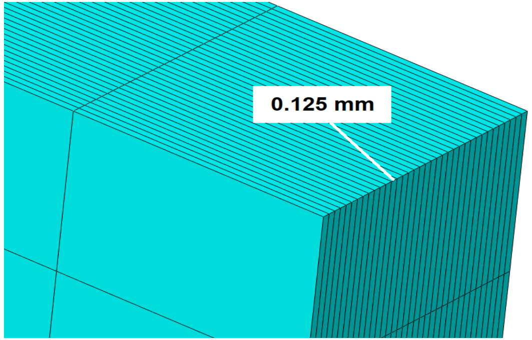
FE model

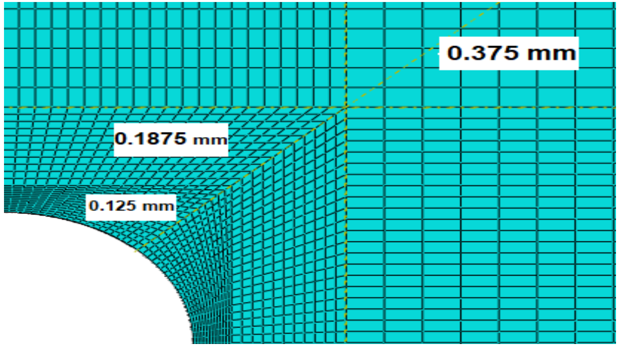
Alexander Nuhn & Christian Mittelstedt

Randspannungskonzentrationen an Kreislöchern in ebenen Laminaten

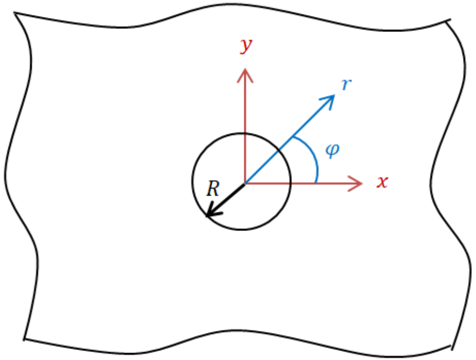
Master Thesis, Technishe Universität Darmstadt **2019**.

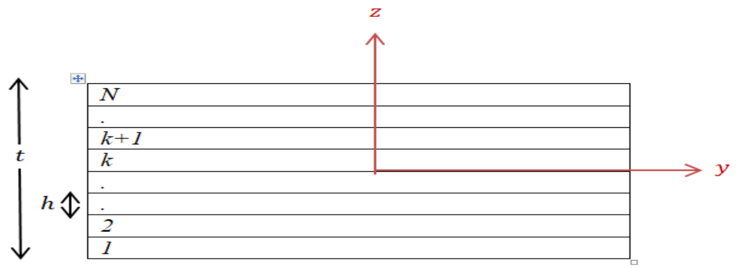






Coordinates, geometry and material properties





Description of the parameter	Symbol	Value	Unit
Hole radius	R	2.5	mm
Hole position	-	central	-
Ply thickness	h	0.125	mm
Total laminate thicknesses	t	0.5	mm
Number of the plies	N	4	-

Description of the parameter	Symbol	Value	Unit
Longitudinal elasticity modulus	E_{11}	35	GPa
Transverse elasticity modulus	E_{22}	9	GPa
Shear modulus	G_{12}	4.7	GPa
Poisson's ratio	ν_{12}	0.28	-
Longitudinal thermal expansion coefficient	α_{11}	5.5×10^{-6}	K^{-1}
Transverse thermal expansion coefficient	α_{22}	2.5×10^{-5}	K^{-1}

Basic formulations

Equilibrium equations in polar coordinate system

$$\frac{\partial \sigma_{rr}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\varphi}^{(k)}}{\partial \varphi}+\frac{\partial \tau_{rz}^{(k)}}{\partial z}+\frac{\sigma_{rr}^{(k)}-\sigma_{\varphi\varphi}^{(k)}}{r}=0$$
$$\frac{\partial \tau_{r\varphi}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\varphi\varphi}^{(k)}}{\partial \varphi}+\frac{\partial \tau_{\varphi z}^{(k)}}{\partial z}+\frac{2\tau_{r\varphi}^{(k)}}{r}=0$$
$$\frac{\partial \tau_{rz}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\varphi z}^{(k)}}{\partial \varphi}+\frac{\partial \sigma_{zz}^{(k)}}{\partial z}+\frac{\tau_{rz}^{(k)}}{r}=0$$

$$\rho=\frac{r-R}{R}\quad \rho\geq 0,\qquad P=\frac{r-R}{h},\qquad \Lambda=\frac{z^{(k)}}{h},\qquad \Gamma=\frac{h}{R}$$

$$\tau_{rz,\Lambda}^{(k)}+\Gamma\left[\sigma_{rr,\rho}^{(k)}+\frac{1}{1+\rho}\left(\tau_{r\varphi,\varphi}^{(k)}+\sigma_{rr}^{(k)}-\sigma_{\varphi\varphi}^{(k)}\right)\right]=0$$
$$\tau_{\varphi z,\Lambda}^{(k)}+\Gamma\left[\tau_{r\varphi,\rho}^{(k)}+\frac{1}{1+\rho}\left(\sigma_{\varphi\varphi,\varphi}^{(k)}+2\tau_{r\varphi}^{(k)}\right)\right]=0$$
$$\sigma_{zz,\Lambda}^{(k)}+\Gamma\left[\tau_{rz,\rho}^{(k)}+\frac{1}{1+\rho}\left(\tau_{\varphi z,\varphi}^{(k)}+\tau_{rz}^{(k)}\right)\right]=0$$

$$\boldsymbol{\sigma}=\boldsymbol{f}^{0}+\boldsymbol{\sigma}^{0}$$

Total stress=boundary region stress+interior region stress

Lekhnitskii's theory adopted with CLPT +perturbation series of Γ

Lekhnitskii's theory of 2D anisotropic elasticity adopted with CLPT

$$f_{ij}(P,\varphi,\Lambda,\Gamma)=\sum_{n=0}^{\infty} f_{ij}^{(n)}(P,\varphi,\Lambda)\Gamma^{n},\qquad f_{ij}^{(n)}=0\ \ for\ \ n<0$$

$$f_{rz,\Lambda}^{n}+f_{rr,\mathrm{P}}^{n}+\mathrm{P}\left(f_{rz,\Lambda}^{n-1}+f_{rr,\mathrm{P}}^{n-1}\right)+f_{r\varphi,\varphi}^{n-1}+f_{rr}^{n-1}-f_{\varphi\varphi}^{n-1}=0$$
$$f_{\varphi z,\Lambda}^{n}+f_{r\varphi,\mathrm{P}}^{n}+\mathrm{P}\left(f_{\varphi z,\Lambda}^{n-1}+f_{r\varphi,\mathrm{P}}^{n-1}\right)+f_{\varphi\varphi,\varphi}^{n-1}+f_{r\varphi}^{n-1}=0$$
$$f_{zz,\Lambda}^{n}+f_{rz,\mathrm{P}}^{n}+\mathrm{P}\left(f_{zz,\Lambda}^{n-1}+f_{rz,\mathrm{P}}^{n-1}\right)+f_{\varphi z,\varphi}^{n-1}+f_{rz}^{n-1}=0$$

For the zeroth-order $n=0$:
(${f_{\varphi\varphi}^{0}}^{(k)}$ has been disappeared)

$${f_{rz,\Lambda}^{0}}^{(k)}+{f_{rr,\mathrm{P}}^{0}}^{(k)}=0$$
$${f_{\varphi z,\Lambda}^{0}}^{(k)}+{f_{r\varphi,\mathrm{P}}^{0}}^{(k)}=0$$
$${f_{zz,\Lambda}^{0}}^{(k)}+{f_{rz,\mathrm{P}}^{0}}^{(k)}=0$$

Simplification

$$f_{ij}^{0\,(k)} = g_{ij}^{(k)}(\Lambda).\,h_{ij}^{(k)}(\mathrm{P})$$

$$f_{rz,\Lambda}^{0\;\;(k)} + f_{rr,\mathrm{P}}^{0\;\;(k)} = 0$$
$$f_{\varphi z,\Lambda}^{0\;\;(k)} + f_{r\varphi,\mathrm{P}}^{0\;\;(k)} = 0$$
$$f_{zz,\Lambda}^{0\;\;(k)} + f_{rz,\mathrm{P}}^{0\;\;(k)} = 0$$

$$\frac{dg_{rz}^{(k)}}{d\Lambda} = g_{rr}^{(k)} \qquad \frac{dg_{zz}^{(k)}}{d\Lambda} = g_{rz}^{(k)} \qquad \frac{dg_{\varphi z}^{(k)}}{d\Lambda} = g_{r\varphi}^{(k)}$$

$$-\frac{dh_{rr}^{(k)}}{d\mathrm{P}} = h_{rz}^{(k)} \qquad -\frac{dh_{rz}^{(k)}}{d\mathrm{P}} = h_{zz}^{(k)} \qquad -\frac{dh_{r\varphi}^{(k)}}{d\mathrm{P}} = h_{\varphi z}^{(k)}$$

$$g_{rr}^{(k)} = G_1^{(k)}$$
$$g_{r\varphi}^{(k)} = G_4^{(k)}$$

$$g_{rz}^{(k)} = G_1^{(k)}\Lambda + G_2^{(k)}$$
$$g_{zz}^{(k)} = \frac{1}{2}G_1^{(k)}\Lambda^2 + G_2^{(k)}\Lambda + G_3^{(k)}$$
$$g_{\varphi z}^{(k)} = G_4^{(k)}\Lambda + G_5^{(k)}$$

$$h_{rr}^{(k)} = H_1^{(k)}e^{-\xi_1 P} + H_2^{(k)}e^{-\xi_1\xi_2 P}$$
$$h_{r\varphi}^{(k)} = H_3^{(k)}e^{-\xi_1 P}$$

$$h_{rz}^{(k)} = H_1^{(k)}\xi_1 e^{-\xi_1 P} + H_2^{(k)}\xi_1\xi_2 e^{-\xi_1\xi_2 P}$$
$$h_{zz}^{(k)} = H_1^{(k)}{\xi_1}^2 e^{-\xi_1 P} + H_2^{(k)}{\xi_1}^2{\xi_2}^2 e^{-\xi_1\xi_2 P}$$
$$h_{\varphi z}^{(k)} = H_3^{(k)}\xi_1 e^{-\xi_1 P}$$

ξ_1 and ξ_2 are decay factors

Boundary and continuity conditions

Considering ${\varepsilon^0_{\varphi\varphi}}^{(k)} = 0$ in the boundary layer region, $f^0_{\varphi\varphi}$ can be determined.

$$\lim_{P\to\infty}\left\{{f^0_{rr}}^{(k)}, {f^0_{r\varphi}}^{(k)}, {f^0_{rz}}^{(k)}, {f^0_{\varphi z}}^{(k)}, {f^0_{zz}}^{(k)}\right\} = 0 \qquad k = 1,2,\dots,N$$

decaying boundary layer stresses ${f^0_{ij}}^{(k)}$ at infinity

$${f^0_{rz}}^{(k)} = {f^0_{\varphi z}}^{(k)} = {f^0_{zz}}^{(k)} = 0$$

disappearing interlaminar stresses on the free surfaces

$$\sigma_{rr}^{(k)} = {f^0_{rr}}^{(k)} + {\sigma^0_{rr}}^{(k)} = 0$$
$$\tau_{r\varphi}^{(k)} = {f^0_{r\varphi}}^{(k)} + {\tau^0_{r\varphi}}^{(k)} = 0$$
$${f^0_{rz}}^{(k)} = 0 \qquad k = 1,2,\dots,N$$

along the hole boundary when $r = R$

$${f^0_{rz}}^{(k)} = {f^0_{rz}}^{(k+1)}$$
$${f^0_{\varphi z}}^{(k)} = {f^0_{\varphi z}}^{(k+1)} \qquad k = 1,2,\dots,N-1$$
$${f^0_{zz}}^{(k)} = {f^0_{zz}}^{(k+1)}$$

through the interface of laminas k and $k+1$

Stress solution for the interior region under temperature change

$$\begin{Bmatrix} \sigma_{rr}^0 \\ \sigma_{\varphi\varphi}^0 \\ \tau_{r\varphi}^0 \end{Bmatrix} = \begin{bmatrix} p^2 & q^2 & 2pq \\ q^2 & p^2 & -2pq \\ -pq & pq & p^2-q^2 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

$$p = \cos\varphi$$

$$q = \sin\varphi$$

solution for 2D thermoelasticity is introduced to determine σ_{xx}, σ_{yy} and τ_{xy}

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}^T = \frac{1}{t}\sum_{k=1}^{N} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \bar{\alpha}_x \\ \bar{\alpha}_y \\ \bar{\alpha}_{xy} \end{Bmatrix}^{(k)} (z_k - z_{k-1}).T$$

$$\bar{\alpha}_x{}^{(k)} = cos^2\theta^{(k)}.\alpha_{11} + sin^2\theta^{(k)}.\alpha_{22}$$
$$\bar{\alpha}_y{}^{(k)} = sin^2\theta^{(k)}\alpha_{11} + cos^2\theta^{(k)}.\alpha_{22}$$
$$\bar{\alpha}_{xy}{}^{(k)} = 2\sin\theta^{(k)}.cos\theta^{(k)}.(\alpha_{11} - \alpha_{22})$$

$$\sigma_{xx} = \frac{\partial^2 A}{\partial y^2}, \sigma_{yy} = \frac{\partial^2 A}{\partial x^2}, \tau_{xy} = -\frac{\partial^2 A}{\partial x \partial y}$$

Airy's stress function $A(x,y)$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

the compatibility relation for 2D problems

Considering Hook's law, the constitutive relation for an anisotropic material is:

$$c_{11}\frac{\partial^4 A}{\partial y^4} - 2c_{16}\frac{\partial^4 A}{\partial x \partial y^3} + (2c_{12} + c_{66})\frac{\partial^4 A}{\partial y^2 \partial y^2} - 2c_{26}\frac{\partial^4 A}{\partial y^3 \partial y} + c_{22}\frac{\partial^4 A}{\partial x^4} = -\alpha_x \frac{\partial^2 T}{\partial y^2} + \alpha_{xy}\frac{\partial^2 T}{\partial x \partial y} - \alpha_y \frac{\partial^2 T}{\partial x^2}$$

$$\alpha_x = c_{11}\chi_x + c_{12}\chi_y + c_{16}\chi_{xy}$$
$$\alpha_y = c_{12}\chi_x + c_{22}\chi_y + c_{26}\chi_{xy}$$
$$\alpha_{xy} = c_{16}\chi_x + c_{26}\chi_y + c_{66}\chi_{xy}$$

$$\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \frac{1}{t}\sum_{k=1}^{N} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \bar{\alpha}_x \\ \bar{\alpha}_y \\ \bar{\alpha}_{xy} \end{Bmatrix}^{(k)} (z_k - z_{k-1})$$

c_{ij} are the components of reduced compliance matrix $[A]^{-1}$

α_x, α_y, and α_{xy} are the thermal expansion coefficients in the global coordinate system

Stress solution for the interior region under temperature change

$$A = A^{(h)} + A^{(p)}$$

$$c_{11}\frac{\partial^4 A^{(h)}}{\partial y^4} - 2c_{16}\frac{\partial^4 A^{(h)}}{\partial x \partial y^3} + (2c_{12} + c_{66})\frac{\partial^4 A^{(h)}}{\partial y^2 \partial y^2} - 2c_{26}\frac{\partial^4 A^{(h)}}{\partial y^3 \partial y} + c_{22}\frac{\partial^4 A^{(h)}}{\partial x^4} = 0$$

$$D_1 D_2 D_3 D_4 A^{(h)} = 0 \qquad D_k = \frac{\partial}{\partial y} - \mu_k \frac{\partial}{\partial x}$$

$$c_{11}\mu^4 - 2c_{16}\mu^3 + (2c_{12} + c_{66})\mu^2 - 2c_{26}\mu + c_{22} = 0$$

$$\mu_1 = \alpha_1 + i\beta_1 \qquad \mu_3 = \alpha_2 + i\beta_2$$

$$\mu_2 = \alpha_1 - i\beta_1 \qquad \mu_4 = \alpha_2 - i\beta_2$$

for symmetric laminate $(c_{16} = c_{26} = 0)$

$$c_{11}\mu^4 + (2c_{12} + c_{66})\mu^2 + c_{22} = 0$$

$$A^{(h)} = 2Re\sum_{k=1}^{2} A_k(Z_k)$$

$$Z_k = x + \mu_k y \qquad k = 1,2$$

$$A = A_1(Z_1) + A_2(Z_2) + \overline{A_1(Z_1)} + \overline{A_2(Z_2)} + A^{(p)}$$

$$\frac{dA}{dz} = \Omega_1(Z_1) + \Omega_2(Z_2) + \overline{\Omega_1(Z_1)} + \overline{\Omega_2(Z_2)} + \Omega^{(p)}$$

Stress solution for the interior region under temperature change

$$A = A_1(Z_1) + A_2(Z_2) + \overline{A_1(Z_1)} + \overline{A_2(Z_2)} + A^{(p)}$$

$$\frac{dA}{dz} = \Omega_1(Z_1) + \Omega_2(Z_2) + \overline{\Omega_1(Z_1)} + \overline{\Omega_2(Z_2)} + \Omega^{(p)}$$

$$\sigma_{xx} = \frac{\partial^2 A}{\partial y^2}, \sigma_{yy} = \frac{\partial^2 A}{\partial x^2}, \tau_{xy} = -\frac{\partial^2 A}{\partial x \partial y}$$

$$\sigma_x = 2Re\{\mu_1{}^2\Omega_1'(Z_1) + \mu_2{}^2\Omega_2'(Z_2)\} + \frac{\partial^2 A^{(p)}}{\partial y^2}$$

$$\sigma_y = 2Re\{\Omega_1'(Z_1) + \Omega_2'(Z_2)\} + \frac{\partial^2 A^{(p)}}{\partial x^2}$$

$$\tau_{xy} = -2Re\{\mu_1\Omega_1'(Z_1) + \mu_2\Omega_2'(Z_2)\} - \frac{\partial^2 A^{(p)}}{\partial x \partial y}$$

$$A^{(p)} = 2Re\big(\Psi. A_t(Z_t)\big)$$

$$\sigma_x = 2Re\{\mu_1{}^2\Omega_1'(Z_1) + \mu_2{}^2\Omega_2'(Z_2)\} + 2Re(\Psi.\mu_t\Omega_t')$$

$$\sigma_y = 2Re\{\Omega_1'(Z_1) + \Omega_2'(Z_2)\} + 2Re(\Psi.\Omega_t')$$

$$\tau_{xy} = -2Re\{\mu_1\Omega_1'(Z_1) + \mu_2\Omega_2'(Z_2)\} - 2Re(\Psi.\mu_t\Omega_t')$$

$$\Psi = \frac{-\alpha_y + \alpha_{xy}\mu_t - \alpha_x\mu_t{}^2}{c_{11}\mu_t{}^4 - 2c_{16}\mu_t{}^3 + (2c_{12} + c_{66})\mu_t{}^2 - 2c_{26}\mu_t + c_{22}}$$

$$\begin{Bmatrix} \sigma_{rr}^0 \\ \sigma_{\varphi\varphi}^0 \\ \tau_{r\varphi}^0 \end{Bmatrix} = \begin{bmatrix} p^2 & q^2 & 2pq \\ q^2 & p^2 & -2pq \\ -pq & pq & p^2 - q^2 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

$$p = \cos\varphi$$

$$q = \sin\varphi$$

Stress solution for the interior region under biaxial tension

Regarding to Lekhnitskii's theory of 2D anisotropic elasticity:

$$\sigma_{rr}^{0} = 2Re[(\sin\varphi - \mu_1\cos\varphi)^2\Psi_1' + (\sin\varphi - \mu_2\cos\varphi)^2\Psi_2'] + \sigma_{xx}^{\infty}\cos^2\varphi + \sigma_{yy}^{\infty}\sin^2\varphi + 2\tau_{xy}^{\infty}\sin\varphi\cos\varphi$$

$$\sigma_{\varphi\varphi}^{0} = 2Re[(\cos\varphi + \mu_1\sin\varphi)^2\Psi_1' + (\cos\varphi + \mu_2\sin\varphi)^2\Psi_2'] + \sigma_{xx}^{\infty}\sin^2\varphi + \sigma_{yy}^{\infty}\cos^2\varphi - 2\tau_{xy}^{\infty}\sin\varphi\cos\varphi$$

$$\tau_{r\varphi}^{0} = 2Re[(\sin\varphi - \mu_1\cos\varphi)(\cos\varphi + \mu_1\sin\varphi)\Psi_1' + (\sin\varphi - \mu_2\cos\varphi)(\cos\varphi + \mu_2\sin\varphi)\Psi_2'] + (\sigma_{yy}^{\infty} - \sigma_{xx}^{\infty})\sin\varphi\cos\varphi + \tau_{xy}^{\infty}(\cos^2\varphi - \sin^2\varphi)$$

σ_{xx}^{∞}, σ_{yy}^{∞}, and τ_{xy}^{∞} are far-field applied stresses

$$\Psi_1' = \frac{-i}{2(\mu_1-\mu_2)(1+i\mu_1)}\{\sigma_{xx}^{\infty} + \sigma_{yy}^{\infty} i\mu_2 + \tau_{xy}^{\infty}(i+\mu_2)\} \times \left\{1 - \frac{r(\cos\varphi + \mu_1\sin\varphi)}{\sqrt{r^2(\cos\varphi + \mu_1\sin\varphi)^2 - R^2(1+{\mu_1}^2)}}\right\}$$

$$\Psi_2' = \frac{i}{2(\mu_1-\mu_2)(1+i\mu_1)}\{\sigma_{xx}^{\infty} + \sigma_{yy}^{\infty} i\mu_1 + \tau_{xy}^{\infty}(i+\mu_2)\} \times \left\{1 - \frac{r(\cos\varphi + \mu_2\sin\varphi)}{\sqrt{r^2(\cos\varphi + \mu_2\sin\varphi)^2 - R^2(1+{\mu_2}^2)}}\right\}$$

$i = \sqrt{-1}$, μ_1 ,and μ_2 are the complex roots can be calculated from the characteristic equation

Stress solution for the boundary layer region

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The in-plane stress components for ply "k" using CLPT and coordinate transformation is:

$$\begin{Bmatrix} \sigma_{rr}^0 \\ \sigma_{\varphi\varphi}^0 \\ \tau_{r\varphi}^0 \end{Bmatrix}^{(k)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{(k)} \begin{Bmatrix} \sigma_{rr}^0 \\ \sigma_{\varphi\varphi}^0 \\ \tau_{r\varphi}^0 \end{Bmatrix}$$

$$[a]^{(k)} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} [\bar{Q}]^{(k)}[b] \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix}^{-1}$$

$[b]$ is the laminate-equivalent anisotropic compliance matrix

$$H_1^{(k)} = H_2^{(k)}\xi_2 = -\frac{\xi_2}{\xi_2-1} a_{12}^{(k)} \sigma_{\varphi\varphi}^0|_{\mathrm{P}=0}$$

$$H_2^{(k)} = \frac{1}{\xi_2-1} a_{12}^{(k)} \sigma_{\varphi\varphi}^0|_{\mathrm{P}=0}$$

$$H_3^{(k)} = -a_{32}^{(k)} \sigma_{\varphi\varphi}^0|_{\mathrm{P}=0}$$

$$G_1^{(k)} = 1$$

$$G_2^{(k)} = \frac{1}{a_{12}^{(k)}} \sum_{j=k+1}^{N} a_{12}^{(j)}$$

$$G_3^{(k)} = \frac{1}{a_{12}^{(k)}} \sum_{j=k+1}^{N} a_{12}^{(j)} \left(j-k-\frac{1}{2}\right)$$

$$G_4^{(k)} = 1$$

$$G_5^{(k)} = \frac{1}{a_{32}^{(k)}} \sum_{j=k+1}^{N} a_{32}^{(j)}$$

considering constant values of ξ_1 and ξ_2 through the laminate due to the interface traction continuity:

$$f_{rr}^{0\,(k)} = H_2^{(k)}\left(e^{-\xi_1\xi_2\mathrm{P}} - e^{-\xi_1\mathrm{P}}\right)$$

$$f_{r\varphi}^{0\,(k)} = H_3^{(k)} - e^{-\xi_1\mathrm{P}}$$

$$f_{rz}^{0\,(k)} = \left(\Lambda + G_2^{(k)}\right) H_2^{(k)} \xi_1\xi_2 \left(e^{-\xi_1\xi_2\mathrm{P}} - e^{-\xi_1\mathrm{P}}\right)$$

$$f_{\varphi z}^{0\,(k)} = \left(\Lambda + G_5^{(k)}\right) H_3^{(k)} \xi_1 e^{-\xi_1\mathrm{P}}$$

$$f_{zz}^{0\,(k)} = \left(\frac{1}{2}\Lambda^2 + G_2^{(k)}\Lambda + G_3^{(k)}\right) H_2^{(k)} {\xi_1}^2 \xi_2 \left(\xi_2 e^{-\xi_1\xi_2\mathrm{P}} - e^{-\xi_1\mathrm{P}}\right)$$

$$f_{\varphi\varphi}^{0\,(k)} = -\left(\bar{S}_{12}^{(k)} f_{rr}^{(k)} + \bar{S}_{23}^{(k)} f_{zz}^{(k)} + \bar{S}_{26}^{(k)} f_{r\varphi}^{(k)}\right) / \bar{S}_{22}^{(k)}$$

$$\sigma_{rr}^{(k)} = \sigma_{rr}^{0\,(k)} + f_{rr}^{0\,(k)}$$

$$\sigma_{\varphi\varphi}^{(k)} = \sigma_{\varphi\varphi}^{0\,(k)} + f_{\varphi\varphi}^{0\,(k)}$$

$$\tau_{r\varphi}^{(k)} = \tau_{r\varphi}^{0\,(k)} + f_{r\varphi}^{0\,(k)}$$

$$\tau_{rz}^{(k)} = f_{rz}^{0\,(k)}$$

$$\tau_{\varphi z}^{(k)} = f_{\varphi z}^{0\,(k)}$$

$$\sigma_{zz}^{(k)} = f_{zz}^{0\,(k)}$$

Solution using energy minimization

- The problem is solved using minimum potential energy principle. It is necessary to calculate ξ_1 and ξ_2 for the zeroth order.
- The external work is dependent to ξ_1 and ξ_2. Therefore it doesn't interfere in the the energy equation.

$$\delta\Pi = \delta\left\{\sum_{k=1}^{N}\frac{1}{2}\int_{-\pi}^{\pi}\int_{0}^{1}\int_{0}^{\infty}\left[\left(\tilde{S}_{11}f_{rr}^{0\,(k)} + \tilde{S}_{13}f_{zz}^{0\,(k)} + \tilde{S}_{16}f_{r\varphi}^{0\,(k)}\right)f_{rr}^{0\,(k)} + \left(\tilde{S}_{13}f_{rr}^{0\,(k)} + \tilde{S}_{33}f_{zz}^{0\,(k)} + \tilde{S}_{36}f_{r\varphi}^{0\,(k)}\right)f_{zz}^{0\,(k)} + \left(\tilde{S}_{16}f_{rr}^{0\,(k)} + \tilde{S}_{36}f_{zz}^{0\,(k)} + \tilde{S}_{66}f_{r\varphi}^{0\,(k)}\right)f_{r\varphi}^{0\,(k)} + \left(\tilde{S}_{44}f_{\varphi z}^{0\,(k)} + \tilde{S}_{45}f_{rz}^{0\,(k)}\right)f_{\varphi z}^{0\,(k)} + \left(\tilde{S}_{45}f_{\varphi z}^{0\,(k)} + \tilde{S}_{55}f_{rz}^{0\,(k)}\right)f_{rz}^{0\,(k)}\right]\times(R + h\mathrm{P})h^2 d\mathrm{P}\,d\Lambda\,d\varphi\right\}$$

$$\frac{\partial\pi}{\partial\xi_1} = {\xi_1}^5R\left(3{\xi_2}^5U_{33} + 3{\xi_2}^4U_{33}\right) + {\xi_1}^5h\left(2{\xi_2}^4U_{33}\right) + {\xi_1}^3R\left[{\xi_2}^4(-2U_{13} - 2U_{36} + U_{44} + 2U_{45} + U_{55}) + {\xi_2}^3(-2U_{13} - 2U_{36} + 2U_{44} + 2U_{45} + U_{55}) + {\xi_2}^2RU_{44}\right] - \xi_1R\left[{\xi_2}^4(U_{11} + 2U_{16} + U_{66}) + {\xi_2}^3(4U_{11} + 6U_{16} + 2U_{66}) + {\xi_2}^2(4U_{11} + 4U_{16} + U_{66}) + \xi_2U_{11}\right] - h\left[{\xi_2}^4(U_{11} + 2U_{16} + U_{66}) + {\xi_2}^3(4U_{11} + 6U_{16} + 2U_{66}) + {\xi_2}^2(8U_{11} + 8U_{16} + U_{66}) + 4\xi_2U_{11} + U_{11}\right] = 0$$

$$\frac{\partial\pi}{\partial\xi_2} = {\xi_2}^6R\left({\xi_1}^5U_{33}\right) + {\xi_2}^5\left(3{\xi_1}^5U_{33}\right) + {\xi_2}^4\left[2{\xi_1}^5RU_{33} + 2{\xi_1}^4hU_{33} + {\xi_1}^3R(-2U_{13} - 2U_{36} + 2U_{45} + U_{55}) - {\xi_1}^2h(-2U_{13} - 3U_{36} + U_{45} + U_{55}) - \xi_1R\left(2U_{11} + 2U_{16} - h(U_{11} + U_{16})\right)\right] + {\xi_2}^3\left[{\xi_1}^3R(-2U_{13} - 2U_{36} + 2U_{45} + U_{55}) + {\xi_1}^2(-2U_{13} - U_{36} + 3U_{45} + U_{55}) - \xi_1R(4U_{11} + 2U_{16}) - h(6U_{11} + 5U_{16})\right] - {\xi_2}^2(3\xi_1RU_{11} + 6hU_{11}) - \xi_2(\xi_1RU_{11} + 4hU_{11}) - hU_{11} = 0$$

32-point Gaussian quadrature formulation

$$U_{11} = \sum_{k=1}^{N} \int_{-\pi}^{\pi} \tilde{S}_{11}^{(k)} {m_1}^2 n.\, d\varphi$$

$$U_{13} = \sum_{k=1}^{N} \int_{-\pi}^{\pi} \tilde{S}_{13}^{(k)} {m_1}^2 n \left(1/6 + 1/2\, G_2^{(k)} + G_3^{(k)}\right).\, d\varphi$$

$$U_{16} = \sum_{k=1}^{N} \int_{-\pi}^{\pi} \tilde{S}_{16}^{(k)} m_1 m_2 n.\, d\varphi$$

$$U_{33} = \sum_{k=1}^{N} \int_{-\pi}^{\pi} \tilde{S}_{33}^{(k)} {m_1}^2 n \left(1/20 + 1/4\, G_2^{(k)} + 1/3\, {G_2^{(k)}}^2 + 1/3\, G_3^{(k)} + {G_3^{(k)}}^2 + G_2^{(k)} G_3^{(k)}\right).\, d\varphi$$

$$U_{36} = \sum_{k=1}^{N} \int_{-\pi}^{\pi} \tilde{S}_{36}^{(k)} m_1 m_2 n \left(1/6 + 1/2\, G_2^{(k)} + G_3^{(k)}\right).\, d\varphi$$

$$U_{66} = \sum_{k=1}^{N} \int_{-\pi}^{\pi} \tilde{S}_{66}^{(k)} {m_2}^2 n.\, d\varphi$$

$$U_{44} = \sum_{k=1}^{N} \int_{-\pi}^{\pi} \tilde{S}_{44}^{(k)} {m_2}^2 n \left(1/3 + G_5^{(k)} + {G_5^{(k)}}^2\right).\, d\varphi$$

$$U_{45} = \sum_{k=1}^{N} \int_{-\pi}^{\pi} \tilde{S}_{45}^{(k)} m_1 m_2 n \left(1/3 + 1/2\left(G_2^{(k)} + G_5^{(k)}\right) + G_2^{(k)} G_5^{(k)}\right).\, d\varphi$$

$$U_{55} = \sum_{k=1}^{N} \int_{-\pi}^{\pi} \tilde{S}_{55}^{(k)} {m_1}^2 n \left(1/3 + G_2^{(k)} + {G_2^{(k)}}^2\right).\, d\varphi$$

$$m_1 = a_{12}^{(k)},\, m_2 = a_{32}^{(k)},\, n = {\sigma_{\varphi\varphi}^{0}}^2 |_{\mathrm{P}=0}$$

Results for pure temprature change (Circumferential directions)

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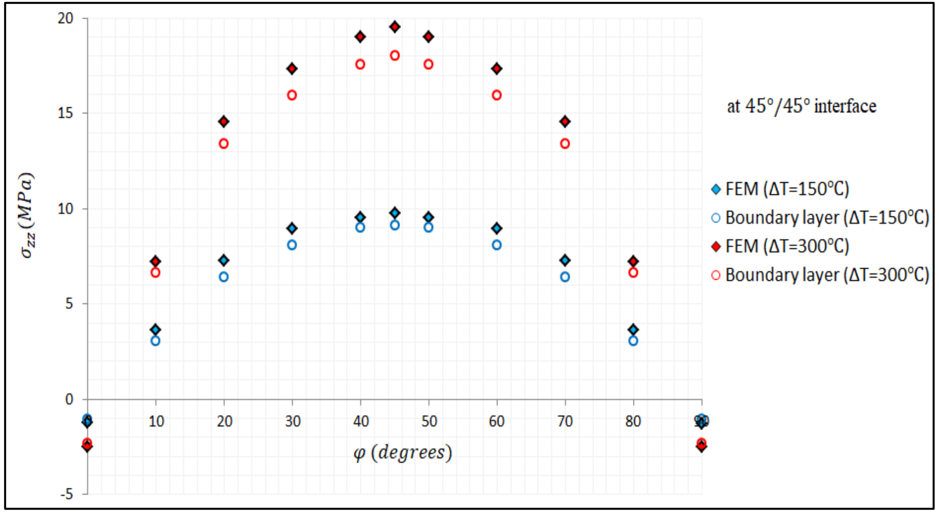
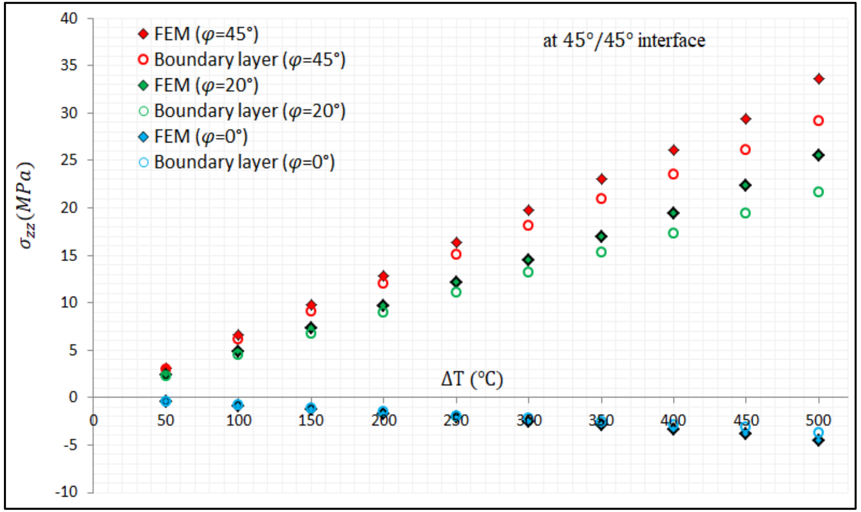


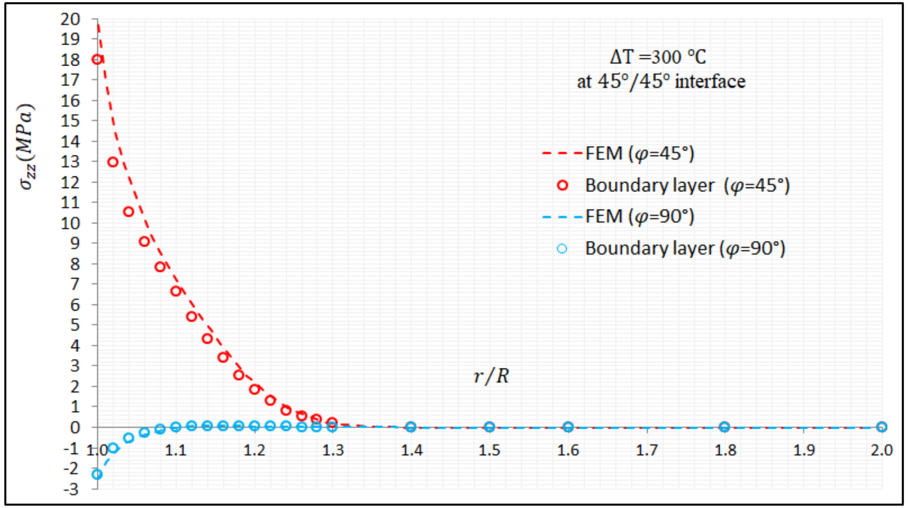
Comparison of the maximum interlaminar peeling stress through the middle interface **along a quarter of the circumferential direction on the edge of the hole for two values of temperature changes** using FEM and boundary layer



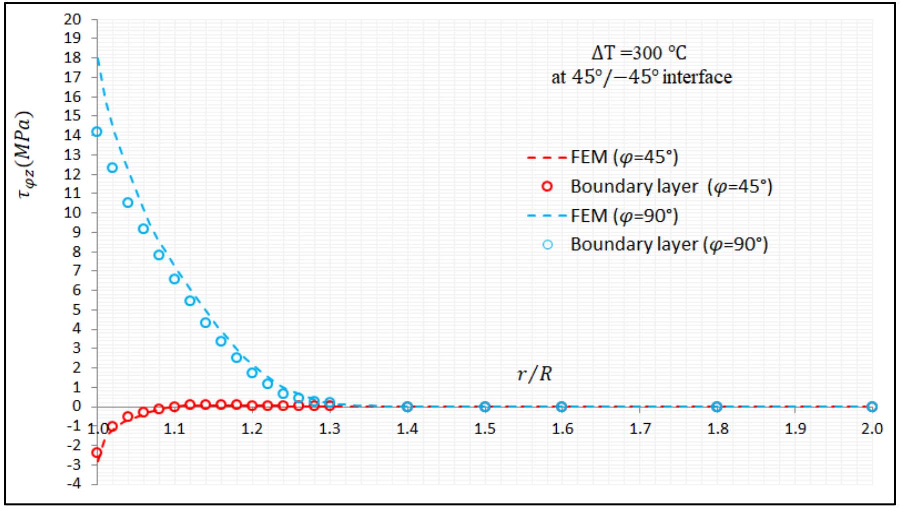
Comparison of the maximum interlaminar peeling stress through the middle interface **at three circumferential positions on the edge of the hole respect to temperature changes** using FEM and boundary layer

Results (radial directions)

Comparison of the maximum **interlaminar peeling stress through the middle interface** along two radial directions for a certain temperature change using FEM and boundary layer



Comparison of the maximum **interlaminar shear stress through 45°/-45° interface** along two radial directions for a certain temperature change using FEM and boundary layer



Results (combined biaxial and temprature change)

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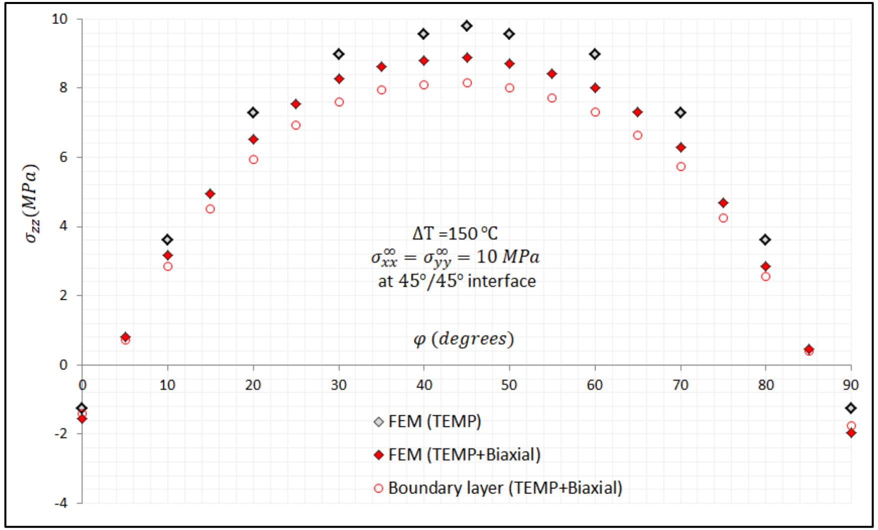
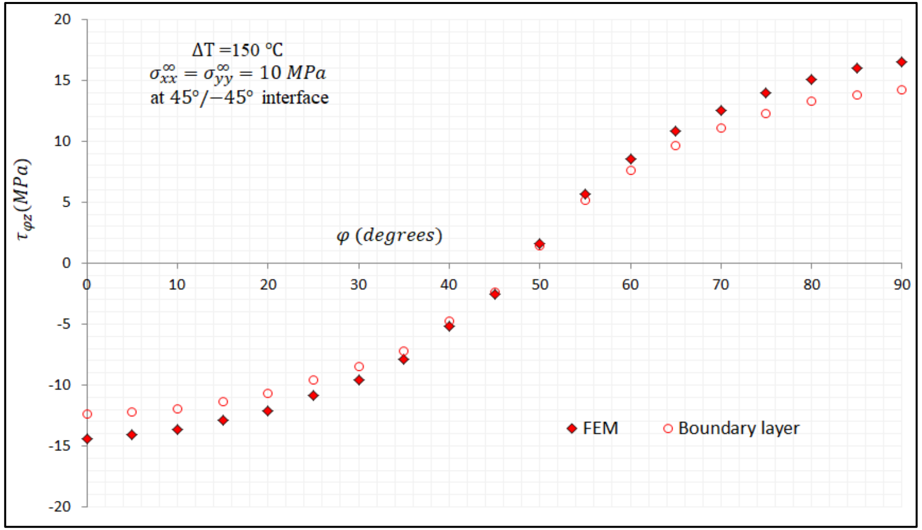
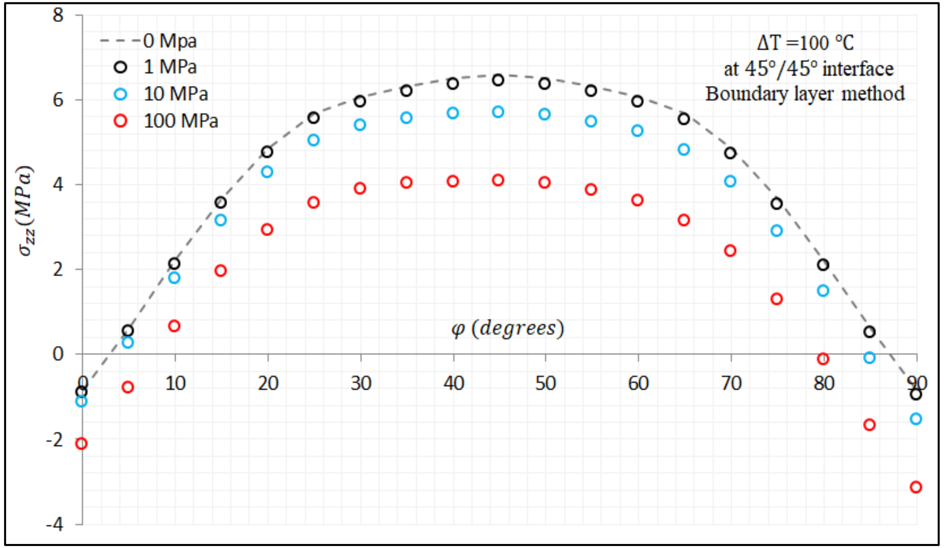


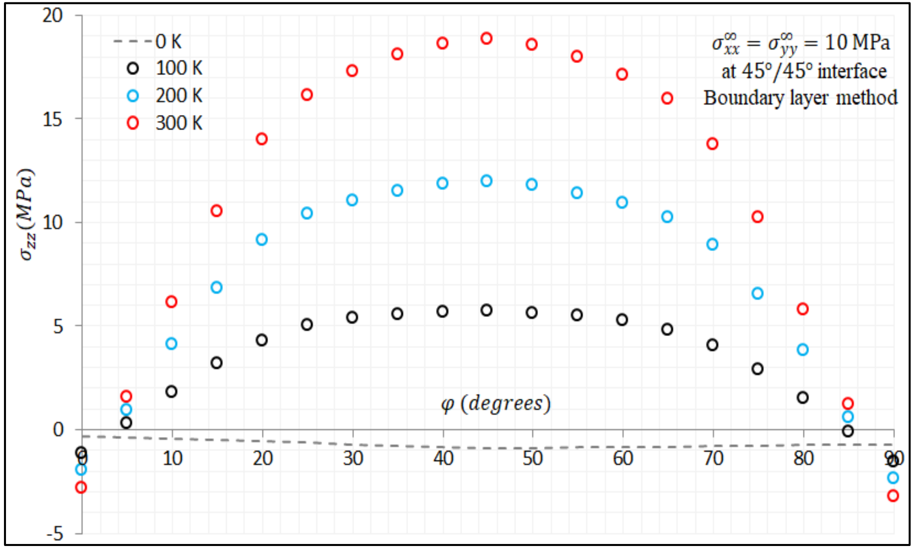
Comparison of the maximum **interlaminar peeling stress through the middle interface along a quarter of the circumferential direction** on the edge of the hole **for pure temperature change** and **combined temperature change with biaxial tension** using FEM and boundary layer



Comparison of the maximum **interlaminar shear stress through 45°/−45° interface** along a quarter of the circumferential direction on the edge of the hole for **combined temperature change and biaxial tension** using FEM and boundary layer



Comparison of the maximum interlaminar peeling stress through the middle interface along a quarter of the circumferential direction on the edge of the hole **for different values of biaxial tension** using boundary layer method



Comparison of the maximum interlaminar peeling stress through the middle interface along a quarter of the circumferential direction on the edge of the hole **for different values of temperature change** using boundary layer method

Conclusion and future work

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- The presented method is capable of predicting peeling and shear interlaminar stresses surrounding a hole in composite lamiantes which are thin and the radius of the hole is small enough in comparison to the dimension of the laminate in both pure temprature change and combined biaxial tension and temprature change.
- With increasing the variation of temprature, the interlaminar peeling stress increases non-lineraly and the discrepancy between the boundary layer method and FEM increases too.
- Along the radial directions at the distance of **1.3R** from the edge of the whole, the peeling and shear interlaminar stresses decay. However in **1.1R**, about 50% of the maximum stresses on the edge vanishes.
- Applying biaxial tension reduces the interlaminar stresses that have been generated by the temprature change.
- Changing the value of temprature is more effective than varying the biaxial tension on the peeling interlaminar stresses.

Future study:

- The effect of moisture change can be investigated by using β instead of α and M instead of T.

$\beta: Moisture\ expansion\ coefficient$

$$M: Moisture\ content = \frac{Mass\ of\ absorbed\ moisture\ in\ the\ material}{Mass\ of\ dry\ material}$$

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Thank you for your attention!

Free-edge effects at circular holes in composite laminates under hygrothermomechanical load

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