A SEMI-ANALYTICAL CONTINUUM DAMAGE MECHANICS MODEL FOR VARIABLE ANGLE TOW COMPOSITE LAMINATES

Dario Campagna[1], Vincenzo Oliveri[2] and Ivano Benedetti[3]

[1] Department of Engineering, University of Palermo, Palermo, Italy, dario.campagna@unipa.it, https://orcid.org/0000-0001-6794-1081
[2] School of Engineering and Bernal Institute, University of Limerick, Limerick, Ireland, Vincenzo.Oliveri@ul.ie, https://orcid.org/0000-0001-7775-4714
[3] Department of Engineering, University of Palermo, Palermo, Italy, ivano.benedetti@unipa.it, https://orcid.org/0000-0003-3755-2768

Keywords: Composite plates, Continuum damage mechanics, Ritz method, Variable angle tow composites

ABSTRACT

In this work, a Ritz formulation for the analysis of damage initiation and evolution in variable angle tow composite plates under progressive loading is presented. The formulation is based on the first order shear deformation theory with von Kármán assumptions and includes the effects of material damage and energy dissipation. The onset of damage is predicted using Hashin's criteria, and a linear softening law is used to describe the degradation of the constitutive relations. The governing equations are obtained by applying the principle of minimum potential energy and using orthogonal polynomials for the approximation of the primary variables. An adaptive multi-domain discretization is introduced to avoid the Gibbs effect and to represent the global response of the structure as a piecewise continuous function. Numerical results are presented to demonstrate the effectiveness of the proposed method in predicting the response of composite structures under different loading conditions and highlight some issues to be addressed by suitable developments of the method.

1 INTRODUCTION

Multilayered composite materials enable the design of lightweight structures with improved stiffness, strength and fatigue properties when compared to metallic structures. For such reason, they find extensive applications in various fields of engineering, including aerospace, naval, and automotive industries. Recent advancements in manufacturing techniques such as automated fibre placement, automated tape laying and additive manufacturing have made it possible to create composite structures with variable mechanical properties. This innovation has led to the development of Variable Angle Tow (VAT) laminates, where the fibre orientation varies throughout the structure based on its position [1].

Modelling progressive damage in composite materials is challenging due to the many damage mechanisms that must be considered. Depending on the idealization's scale, damage can be modelled in various ways, from the micro- to the macro-scale. Using a micromechanical approach, damage in the form of cracks is represented using representative volume elements (RVSs) [2-4]. Instead, at macro-scale level, damage is represented as a hard discontinuity [5, 6]. At an intermediate scale, namely the meso-scale level, the individual plies are represented as homogenous and, among the different approaches that have been considered, one of the most common theories used to model the damage process is the Continuum Damage Mechanics (CDM).

CDM models are based on the works done by Matzenmiller and Ladeveze [7, 8], among others, where damage is represented as a progressive loss of material stiffness. Within the CDM framework, different models have been developed and used in finite element (FE) approaches [9, 10]. In addition to FE-based analysis methods, single domain meshless approaches, such as the Ritz method, have been shown to be competitive, especially when dealing with smeared damaged zones [11].

However, in few cases, damage tends to concentrate in a narrow region due to specific loading conditions or initial imperfection [12, 13]. Within a classical single domain Ritz approach, this localization of damage can introduce spurious effects when reconstructing the damaged state, commonly referred to as the Gibbs effect, which can result in non-physical responses. Therefore, designers and engineers need to be mindful of the constraints and limitations associated with various modeling and computational tools when predicting the structural behavior of composite materials and components. Although the Ritz method offers several advantages, there is limited literature considering this approach to investigate damage initiation and evolution. Existing works on this topic often employ overly simplified damage models that provide a binary representation of damage [14], which is more suitable for identifying damage initiation rather than capturing damage evolution. Therefore, these approaches tend to be overly conservative.

The key novelty and objective of this study is to develop an adaptive multi-domain Ritz method that overcomes spurious numerical effects while retaining the advantages of the Ritz approach, such as reduced degrees of freedom compared to FE models, while ensuring a highly accurate description of the damage process and a physically meaningful response.

2 METHODS

In this section, the key items of the formulation are briefly discussed.

2.1 Kinematics and strain displacement relationship

In the present formulation, the kinematics is based on the First order Shear Deformation Theory (FSDT) with von Kármán assumptions accounting for geometric non-linearities. Referring to a Cartesian coordinate system $x_1 x_2 x_3$ with the x_3 axis directed along the thickness the displacements are given by

$$\boldsymbol{d} = \boldsymbol{u} + x_3 \boldsymbol{L}\boldsymbol{\vartheta} + \overline{\boldsymbol{w}}, \tag{1}$$

where,

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2}$$

In Eq. (1) the vectors $\boldsymbol{u} = \{u_1, u_2, u_3\}^T$ and $\boldsymbol{\vartheta} = \{\vartheta_1, \vartheta_2, \vartheta_3\}^T$ collect the reference plane displacement and rotation components, respectively, while , $\overline{\boldsymbol{w}} = \{0,0,\overline{w}\}^T$ is the vector introducing a prescribed imperfection.

In the framework of small strains and moderate rotations, geometric non-linearities are assumed in the von-Karman sense; the kinematic is described by the strain vector $\boldsymbol{e}$ which is partitioned into the in-plane and out-of-plane components, denoted by subscript *p* and *n* respectively:

$$\boldsymbol{e} = \left\{\boldsymbol{e}_p, \boldsymbol{e}_n\right\}^T. \tag{3}$$

The strain-displacement relations are therefore given by

$$\boldsymbol{e}_p = \boldsymbol{\mathcal{D}}_p \boldsymbol{u} + \frac{1}{2}\left(\boldsymbol{\mathcal{D}}_p \otimes u_3\right)\boldsymbol{\mathcal{D}}_n \boldsymbol{u} + x_3 \boldsymbol{\mathcal{D}}_p \boldsymbol{L}\boldsymbol{\vartheta} + \left(\boldsymbol{\mathcal{D}}_p \otimes \overline{\boldsymbol{w}}\right)\boldsymbol{\mathcal{D}}_n \boldsymbol{u} \tag{4}$$

$$\boldsymbol{e}_n = \boldsymbol{\mathcal{D}}_n \boldsymbol{u} + \boldsymbol{L}\boldsymbol{\vartheta}, \tag{5}$$

where $\boldsymbol{\mathcal{D}}_p$ and $\boldsymbol{\mathcal{D}}_n$ are differential operator and the symbol $\otimes$ denotes the Kronecker product [15].

2.2 Constitutive relations

Considering a plane stress assumption for the k-th lamina, the following relations hold

$$\boldsymbol{\sigma}_p = \boldsymbol{Q}_p \boldsymbol{e}_p, \quad \boldsymbol{\sigma}_n = \boldsymbol{Q}_n \boldsymbol{e}_n, \tag{6}$$

in which the elastic coefficient matrices $\boldsymbol{Q}_p$ and $\boldsymbol{Q}_n$ depend on the local fiber orientation for each ply [15].

Following a classical CDM approach, the in-plane stiffness coefficients are functions of the evolution of the damage. Namely, the onset of the damage is predicted using the Hashin's criteria [16] and four damage indices are computed and used for the degradation of the constitutive relations at the ply level, in fiber and matrix directions, for both tension and compression. The generic damaged in-plane stiffness matrix in the orthotropic material reference system reads as [7]

$$\boldsymbol{C} = \begin{bmatrix} (1-\omega_1)E_1 & (1-\omega_1)(1-\omega_2)\nu_{21}E_1 & 0 \\ (1-\omega_1)(1-\omega_2)\nu_{12}E_2 & (1-\omega_2)E_2 & 0 \\ 0 & 0 & (1-\omega_6)G_{12} \end{bmatrix}, \qquad (7)$$

with $D = 1 - (1-\omega_1)(1-\omega_2)\nu_{12}\nu_{21}$. In Eq. (8), ω_1, ω_2 and ω_6 are the longitudinal (fiber–dominated), transverse (matrix–dominated) and shear damage indices respectively.

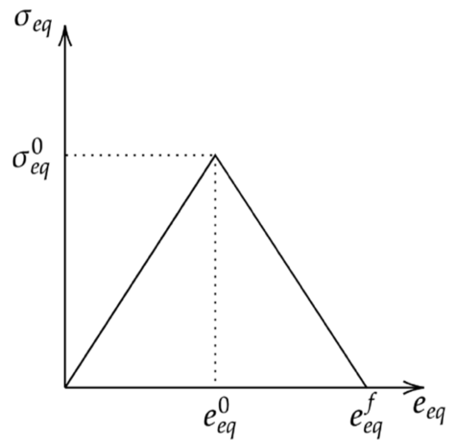

Figure 1: Linear softening constitutive law.

The evolution of damage is based on a linear softening law (see Fig. 1), and it is assessed by means of equivalent strains. The area under the stress-strain curve in Fig. (1) corresponds to the energy dissipated per unit volume in the loading process until failure,

$$g = \int_0^{e_{eq}^f} \sigma_{eq} de_{eq}. \qquad (8)$$

As proposed in Refs. [11, 14, 17], it is possible to provide the equivalent strain ratio α as an input parameter to define the final equivalent displacement e_{eq}^f following the relation,

$$e_{eq}^f = \alpha e_{eq}^0. \qquad (9)$$

Note, this approach is suitable for loading cases leading to a spread of the damage in a well-defined area which must be independent from the polynomial expansion used in the Ritz approximation scheme. If the damage tends to localize in a narrow band due to the increasing of the polynomial order, the boundary value problem becomes ill-posed, and it is necessary to introduce an energy regularization scheme. Moreover, the localization phenomenon can cause the numerical model to suffer from other spurious effects when reconstructing the damage state, known as the Gibbs effect, which leads towards a non-physical response. These issues are due to the coupling of a classical single domain Ritz approach and the localization phenomenon.

To overcome such limitations, in the present work an adaptive multi-domain discretization is proposed. Starting with a single domain discretization, the onset of the damage is monitored at specific sampling points corresponding to the domain integration points. When damage initiation is triggered, the adaptive multi-domain is activated to refine the discretization in the neighborhood of the damaged zone. This multi-domain refinement allows to represent the global response of the structure, in terms of strains, as a piecewise continuous function removing the Gibbs effect arising in a single domain discretization. Some numerical test cases showing the suppression of the Gibbs effect are presented in the next section.

To ensure a consistent energy dissipation with respect to the Ritz kinematic approximation during the failure process, a smeared approach is adopted by distributing the fracture energy over an area associated with the gauss integration points of the domain of interest [9, 18, 19]. Referring to Fig. 1, the fracture energy dissipated per unit area can be computed as

$$G_c = \int_0^w \int_0^{e_{eq}^f} \sigma_{eq}\, de_{eq}\ dx = w\frac{1}{2}\sigma_{eq}^0 e_{eq}^f, \tag{10}$$

where w is the square root of the area associated with a generic gauss point.

2.3 Governing equations

Using orthogonal polynomials for the approximation of the primary variables [15] and applying the principle of the minimum potential energy, the governing equation can be written as, [11]

$$(\boldsymbol{K}_0 + \overline{\boldsymbol{K}}_0 + \boldsymbol{K}_1 + \boldsymbol{K}_2 + \overline{\boldsymbol{K}}_1 + \boldsymbol{R})\boldsymbol{X} = \boldsymbol{F}_D + \boldsymbol{F}_L. \tag{11}$$

where, $\boldsymbol{X}$ is the vector collecting the unknown Ritz coefficients, $\boldsymbol{K}_0, \boldsymbol{K}_1, \boldsymbol{K}_2, \overline{\boldsymbol{K}}_0, \overline{\boldsymbol{K}}_1$ are the stiffness matrices of the problem. In Eq. (11) the subscript 0 indicates linear terms, while subscripts 1 and 2 refer to the geometric non-linear terms and the over-bar refers to prescribed initial imperfections. Furthermore, in Eq. (11) $\boldsymbol{R}$ is the penalty matrix related to the enforcement of the BCs thorough a penalty approach and the vectors $\boldsymbol{F}_D$ and $\boldsymbol{F}_L$ collect the discrete terms associated with the external loads.

To solve the non-linear problem given in Eq. (11) the incremental form of the governing equation may be expressed as [11]

$$\boldsymbol{R}\Delta\boldsymbol{X} + \Delta\left[\left(\boldsymbol{K}_t^{\boldsymbol{g}} + \boldsymbol{K}_t^{\boldsymbol{d}}\right)\boldsymbol{X}\right] = \Delta\boldsymbol{F}_D + \Delta\boldsymbol{F}_L. \tag{12}$$

where, $\boldsymbol{K}_t^{\boldsymbol{g}}$ and $\boldsymbol{K}_t^{\boldsymbol{d}}$ are the tangent stiffness matrices related to geometric non-linearities and initial imperfections, and the damage evolution, respectively. The developed semi-analytical model has been implemented in an efficient analysis tool, where Eq. (11) is solved through Newton-Raphson numerical schemes or the arc-length method to properly handle contingent snap-back phenomena. The reader is referred to [11] for more details about the matrices appearing in Eq. (11) and Eq. (12).

3 RESULTS

To assess the capability offered by the developed analysis tool, preliminary results are herein reported.

3.1 VAT under compressive loading

The first test analyzes some VAT composite plates under compression loading, considering the presence of moderate strains in the von-Karman sense. Square plates with sides of size $a = b = 250\,mm$ are considered. Four lay-ups, namely $[90 \pm \langle 0|75\rangle]_{3S}$, $[0 \pm \langle 0|15\rangle]_{3S}$, $[0 \pm \langle 0|45\rangle]_{3S}$, $[0 \pm \langle 45|0\rangle]_{3S}$ are investigated. They consist of 12 constant thickness plies, each 0.27 mm thick. The material properties for each orthotropic layer, along the fibers and transverse directions, are reported in Table 1.

Elastic property	Value	Strength property	Value
E_1	181.00 GPa	X_T	1500.0 MPa
E_2	10.27 GPa	X_C	1200.0 MPa
G_{ij}	7.17 GPa	Y_T	40.0 MPa
ν_{12}	0.34	Y_C	176.0 MPa
		S_L	68.0 MPa

Table 1: Material properties for compressive test, where $G_{ij} = G_{23} = G_{13} = G_{12}$.

In this case, the plates do not experience any localization of the damage with respect of the polynomial expansion, without exhibiting the Gibbs's effect.

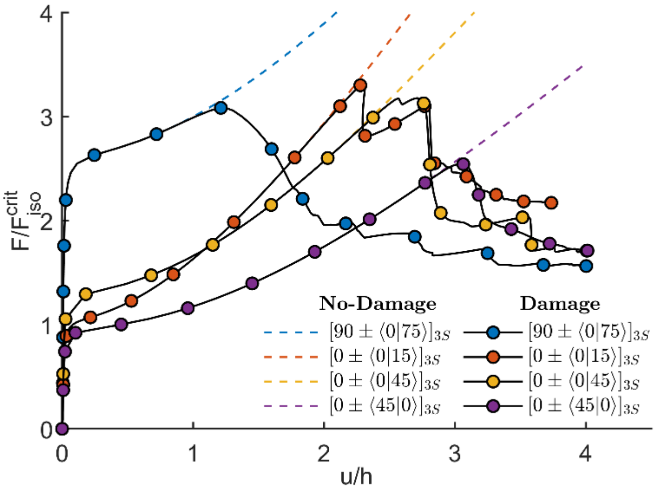

Figure 2: Post-buckling result for different VAT laminates under compression loading.

Fig. 2 shows the obtained post-buckling results of the analyzed VAT composite plates using a single domain approach. The applied load and out-of-plane displacement are normalized with respect to the critical buckling load of a quasi-isotropic layup and the plate thickness, respectively. These results show that, for the considered layups, the laminate achieving the greatest buckling load, is also the one that fails at the lowest out-of-plane displacement. This shows an example on how the developed tool may be used to efficiently investigate damage characteristics of VAT laminates and find trade-off design solutions.

3.2 Damage localization

After considering a test case where the damage is distributed over a well-defined area, some tests of damage localization are reported, to assess the occurrence of the Gibbs phenomenon. To illustrate the problem, a rectangular unidirectional composite plate subjected to uniaxial tension is considered, as shown in Fig. 3. Moreover, a narrow band of material along the x_2 axis has strength lower than the rest of the plate, to artificially induce the onset of the damage.

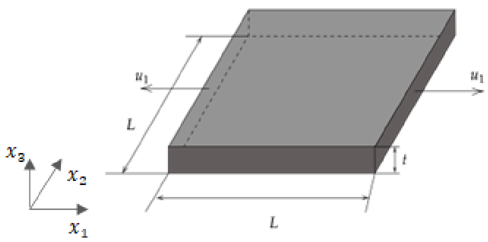

Figure 3: Schematic representation of rectangular plate under uniaxial tension.

Fig. 4 shows the contour plot of the damage state in fiber direction of the plate obtained with four different polynomial expansions, where the emergence of the Gibbs phenomenon is clearly highlighted.

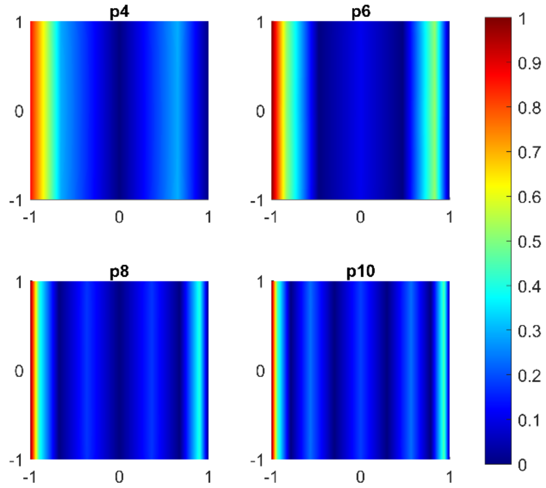

Figure 4: Gibbs effect on damage contour plots of the plate under uniaxial tension.

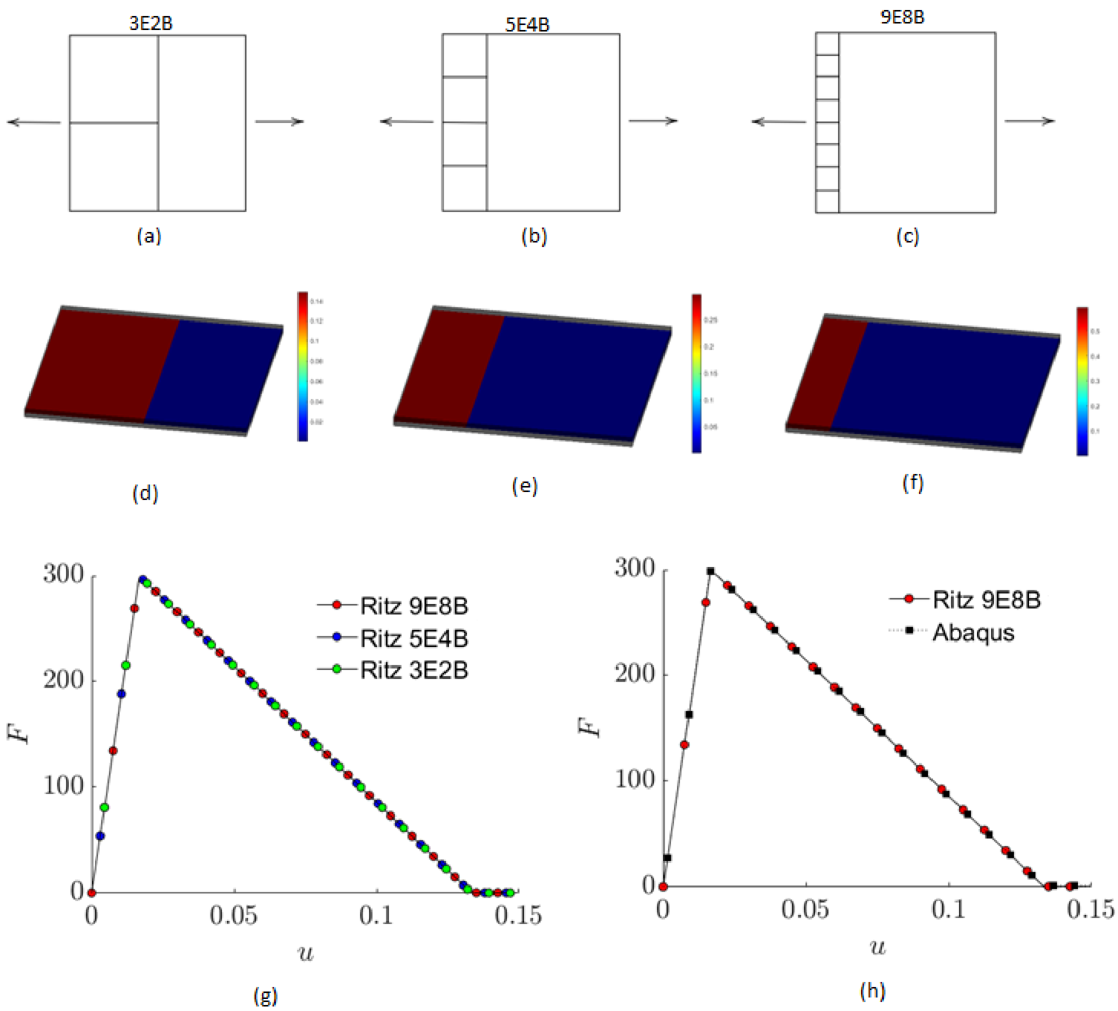

Figure 5: (a)-(c) multi-domain discretization used. (d)-(f) damage plot of each discretization. (g) Comparison of result for different discretization. (h) comparison of results with Abaqus.

To address this issue, within this work, different approaches (e.g., the use of filters to smooth out the Gibbs ripples) have been considered, which have shown to be beneficial, but not fully effective. Hence, the discretization of the entire plate using an adaptive multi-domain representation has been considered. Figs. (5a–c) show three different multi-domain discretization of the plate subjected to uniaxial tension. The generalized displacement of the patches in the narrow band, where the damage spread, are approximated using first order polynomials, whilst in the bigger patch a higher polynomial degree is maintained to avoid losing accuracy. The use of a first order polynomials is crucial for having a constant representation of the strain state in the neighborhood of the damage which lead to a constant damage spread inside the considered damaged patch. Under this approach, the corresponding damage plots are reported in Figs. (5d–f), which show how the Gibbs effect is completely removed. Moreover, Fig. 5(g) shows the results in term of force-vs-displacements, which confirms the independence from the level of multi-domain discretization used, thus validating the objectivity of the response. Finally, the solution of the present method is compared with FE results obtained with the ABAQUS built-in CDM model in Fig. (5h). The comparison of the results shows excellent agreement with established FE analysis methods.

3.3 Crack propagation

To assess the capabilities of the developed adaptive muti domain Ritz method, the damage evolution in a composite unidirectional lamina with a pre-existing crack under tensile load has been considered. The plate was modelled taking advantage of the symmetry to reduce the overall number of degrees of freedom. The geometry and boundary conditions of this test case are reported in Fig. 6.

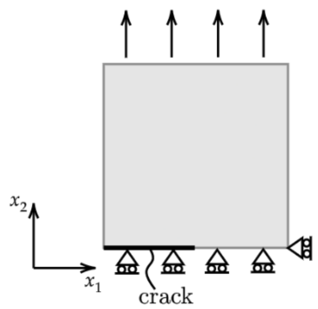

Figure 6: Geometry and boundary condition for unidirectional lamina with a pre-existing crack

The whole plate domain was divided into 3 sub-domains, as reported in Fig. 7(a), where the sub-domain (1) was used to model the pre-existing crack by setting all damage indices $\omega_i = 1$.

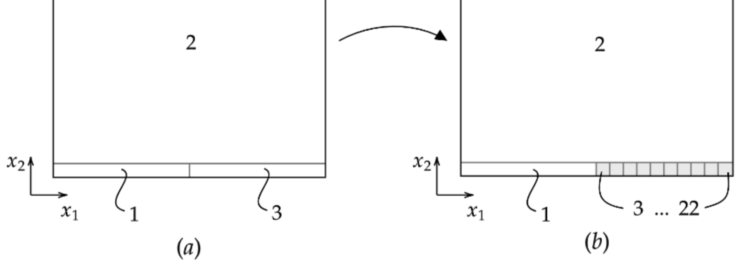

Figure 7: (a) Initial discretization; (b) adaptive multi-domain discretization.

During the incremental loading, the damage starts developing in the vicinity of the crack tip. The damage propagation path is kwon a priori, and the adaptive refinement is activated using the discretization of Fig 7(b) which consist in dividing the sub-domain (3) in 20 new subdomains to describe the damage propagation zone. The convergence of the solution with respect to the polynomial order has been assessed by considering the total reaction force and the displacement in the x_2 direction, as shown in Fig. 8(a). The analysis shows that, for the considered case, convergence is quickly achieved using a polynomial order $p = 6$ for the undamaged part. These results have been compared to results obtained with ABAQUS where a converged mesh of 20x20 finite elements has been used. As shown in Fig 8(b), a very good agreement of the results is achieved, which validate the proposed method. Moreover, the reduction of degree of freedom used for the analysis is noticeable: the proposed model with the polynomial expansion $p = 6$ has a total number of degrees of freedom equal to 1068 whilst the FE analysis has 2646 DOF.

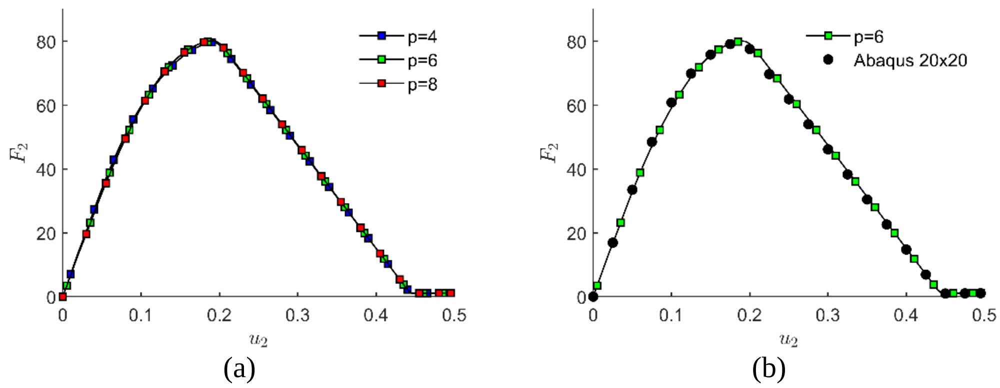

Figure 8: (a) Convergence analysis of the pre-cracked plate; (b) comparison of result with Abaqus

4 CONCLUSIONS

In this work, an adaptive multi-domain Ritz approach for the analysis of progressive failure of composite variable stiffness plates has been developed.

The present study adopts a First Order Shear Deformation theory with non-linear von Karman strains assumptions for representing geometrically non-linear deformations. A Continuum Damage Mechanics framework is employed for capturing the initiation and evolution of damage. Such multi-domain approach has been shown effective in removing the Gibbs effect that arises in single domain Ritz discretization when the damage tends to localize in narrow bands.

Being characterized by a considerably lower computational cost with respect to commercially available finite element procedures, the developed analysis tool may be useful for identifying the operational limits and providing valuable insights to the designer for the analysis of VAT plates.

Also, this set the basis for future developments which could include the study of the impact induced damage analysis for VAT composite plates.

ACKNOWLEDGEMENTS

The Authors gratefully acknowledge the support of the PON Ricerca e Innovazione 2014-2020 – Fondo Sociale Europeo, Azione I.1 “Dottorati Innovativi con caratterizzazione Industriale” – Ciclo XXXVI (CUP: B73D20005010001 – Scholarship ID: DOT20KTEXX). I.B. also acknowledges the support of the Department of Engineering of the University of Palermo through the grant *Premio gruppo di ricerca 2020*.

REFERENCES

[1] Z. Gürdal, BF Tatting and CK Wu, Variable stiffness composite panels: Effects of stiffness variation on the in-plane and buckling response, *Composite Part A-Applied Science and Manufacturing*, **39 (5)**, 2008, pp. 911-922 (doi: 10.1016/j.compositesa.2007.11.015).

[2] C. González and J. Llorca, Mechanical behavior of unidirectional fiber-reinforced polymer under transverse compression: microscopic mechanism and modeling, *Composites Science and Technology*, **67** (13), 2007, pp. 2795-2806 (doi: 10.1016/j.compscitech.2007.02.001).

[3] I. Benedetti., V. Gulizzi and A. Milazzo, A microstructural model for homogenisation and cracking of piezoelectric polycrystals, *Computer Methods in Applied Mechanics and Engineering*, ***357***, 2019, p. 112595 (doi: 10.1016/j.cma.2019.112595).

[4] R. Manno, W. Gao and I. Benedetti, Engineering the crack path in lattice cellular materials through bio-inspired micro-structural alterations, *Extreme Mechanics Letters*, ***26***, 2019, pp. 8-17 (doi: 10.1016/j.eml.2018.11.002).

[5] A. Milazzo, I. Benedetti and V. Gulizzi, A single-domain Ritz approach for buckling and post-buckling analysis of cracked plates, *International Journal of Solids and Structures*, ***159***, 2019, pp. 221-231 (doi: 10.1016/j.ijsolstr.2018.10.002).

[6] A. Milazzo and V. Oliveri, Investigation of buckling characteristics of cracked variable stiffness composite plates by an eXtended Ritz approach, *Thin-Walled Structures*, **163,** 2021, p. 107750 (doi: 10.1016/j.tws.2021.107750).

[7] A. Matzenmiller, J. Lubliner and R. Taylor, A constitutive model for anisotropic damage in fiber-composites, *Mechanics of Materials*, **20**, 1995, pp. 125–152 (doi: 10.1016/0167-6636(94)00053-0).

[8] P. Ladeveze and E. LeDantec, Damage modelling of the elementary ply for laminated composites, *Composites Science and Technology*, **43 (3)**, 1992, pp. 257-267 (doi: 10.1016/0266-3538(92)90097-M).

[9] I. Lapczyk and J. A. Hurtado, Progressive damage modeling in fiber-reinforced materials, *Composites Part A: Applied Science and Manufacturing*, **38 (11)**, 2007, pp. 2333–2341 (doi: 10.1016/j.compositesa.2007.01.017).

[10] P. Maimí, P. Camanho, J. Mayugo and C. Dávila, A continuum damage model for composite laminates: Part I - constitutive model, *Mechanics of Materials*, **39**, 2007, pp. 897-908 (doi: 10.1016/j.mechmat.2007.03.005).

[11] D. Campagna, A. Milazzo, I. Benedetti and V. Oliveri, A non-linear Ritz method for progressive failure analysis of variable angle tow composite laminates, *Mechanics of Advanced Materials and Structures*, **30 (5)**, 2023, pp. 995-1008 (doi: 10.1080/15376494.2022.2134951).

[12] I. Benedetti, H. Nguyen, R. A. Soler-Crespo, W. Gao, L. Mao, A. Ghasemi, J. Wen, S. Nguyen and H. D. Espinosa, Formulation and validation of a reduced order model of 2D materials exhibiting a two- phase microstructure as applied to graphene oxide, *Journal of the Mechanics and Physics of Solids*, **112**, 2018, pp. 66-88 (doi: 10.1016/j.jmps.2017.11.012).

[13] M. Jirásek and D. Rodrigue, Localization analysis of nonlocal models with damage-dependent nonlocal interaction, *International Journal of Solids and Structures*, **174**, 2019, pp. 1-17 (doi: 10.1016/j.ijsolstr.2019.06.011).

[14] Q. J. Yang and B. Hayman, Simplified ultimate strength analysis of compressed composite plates with linear material degradation, *Composites Part B: Engineering*, **69**, 2015, pp. 13–21 (doi: 10.1016/j.compositesb.2014.09.016).

[15] A. Milazzo and V. Oliveri, Buckling and Postbuckling of Stiffened Composite Panels with Cracks and Delaminations by Ritz Approach, *AIAA Journal*, **53 (3)**, 2017, pp 965-980 (doi: 10.2514/1.J055159).

[16] Z. Hashin, Failure Criteria for Unidirectional Fiber Composites, *Journal of Applied Mechanics* **47 (2)**, 1980, pp. 329–334 (doi: 10.1115/1.3153664).

[17] X. Deng, A. Korobenko, J. Yan, and Y. Bazilevs, Isogeometric analysis of continuum damage in rotation-free composite shells, *Computer Methods in Applied Mechanics and Engineering*, **284**, 2015, pp. 349–372 (doi: 10.1016/j.cma.2014.09.015).

[18] Z. P. Bažant and B. H. Oh, Crack band theory for fracture of concrete, *Materiaux et construction*, **16 (3)**, 1983, pp. 155–177 (doi: 10.1007/BF02486267).

[19] M.H. Nagaraj, J. Reiner, R. Vaziri, E. Carrera and M. Petrolo, Progressive damage analysis of composite structures using higher-order layer-wise elements, *Composites Part B: Engineering*, **190**, 2020, p. 107921 (doi: 10.1016/j.compositesb.2020.107921).