AUTOMATED RITZ METHOD FOR THE ANALYSIS OF LAMINATED ANISOTROPIC PLATES

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ABSTRACT

Most of the available functional form solutions of anisotropic plates are available for cases involving simple boundary conditions such clamped and simply supported edges. The proposed method which is based on the well-known Ritz method offers an alternative semi-analytical solution in terms of simple polynomials. As per Ritz method, the trial functions are required to satisfy the essential (geometric) boundary conditions and therefore free edges can be accommodated by the method. However, large number of polynomials is needed in order to get the required convergence which cannot be handled by the standard Ritz method due to the huge number of symbolic integrals to be computed. This difficulty is overcome by casting the method in a matrix form the elements of which are derived using indicial notation and integrated using a symbolic Mathematica code. The code is capable of accommodating a large number of polynomials as required by the accuracy and convergence of the solution. Several numerical examples are presented to verify the accuracy and efficiency of the proposed method.

1 INTRODUCTION

Laminated anisotropic plates are increasingly being used in structural applications due to their high strength-to-weight, stiffness-to-weight ratios, durability, and design flexibility. However, the analysis of laminated anisotropic plates exhibits complexity owing to varying the material properties of such plates through the 3D space. The most popular application of anisotropic plates is the laminated composite plates made of fibers reinforced laminas where the anisotropy level depends on the fibers orientation angle and the stacking sequence [1]–[4]. The classical laminate plate theory (CLPT), which neglects the out-of-plane strains effect, was introduced for the bending analysis of composite laminated plates [3], [5]. The CLPT was proven to be adequate for predicting the behaviour of thin laminated plates as compared to the theory of elasticity [6]–[8]. The governing differential equations of an angle-ply composite plate having two opposite simply supported edges had been reduced to ordinary differential equations by using the Fourier series expansion of load and deflections [9], [10]. Ashton presented the governing differential equations [11] and energy formulation [12], [13] for the analysis of linear elastic bending of anisotropic composite plate without considering the in-plane displacements of the midplane. Also, Wang et al. [14] neglected the in-plane displacements of the midplane in the formulation of strip element method (SEM) to solve the bending problem of symmetric laminated composite plates in the sense of CLPT. Numerical methods were also implemented in solving anisotropic plate bending problem such as finite element method [15], [16], dynamic relaxation method [17], [18], boundary element method [19].

Although, the linear bending behaviour of composite laminated plates has been extended to account for large deflection [20]–[26], the given examples addressed special cases of isotropic, orthotropic, antisymmetric and symmetrical angle-ply laminates. Furthermore, some of them reported the results of deflection only.

Energy methods [27] have the advantages of providing the solution in a continuous functional form which is more suitable for optimization and design purposes. Ritz method, in particular, has a further advantage over other energy methods in handling plate with mixed boundary conditions [28]–[31]. However, as with any approximate method, Ritz method is not free from disadvantages. Particularly when applied to the present problem, the direct application of the method involves symbolic integration of many and lengthy expressions due to the aspect of material anisotropy. To overcome these computational efforts, and at the same time preserve the advantage of the Ritz method in obtaining the solution in a functional form, an automated systematic approach was proposed for the large deflection of anisotropic composite plates [32].

In this paper, the automated Ritz method is proposed for the linear analysis of laminated anisotropic plates under lateral load with mixed boundary conditions. First, the indicial notations are employed to derive the constitutive equations and the resulted equations from the minimization of potential energy. The obtained equations are cast it in a matrix-form which are, then, encoded in a Mathematica program that automates the solution for arbitrary trial approximation functions. Using the proposed approach, the solution starts by approximating the plate displacements u, v, and w by trial functions satisfying the geometric boundary conditions (displacements/slopes) with unknown constants. These expressions are then substituted in the matrix form to solve for the unknown constant and, hence, the complete solution of the problem. To test the adequacy and accuracy of the proposed formulation and the developed code, two numerical examples of laminated angle ply plates are solved, and their solutions are compared with the finite element results obtained by ABAQUS. The results of the comparison confirm that the proposed method is accurate and efficient in predicting the bending behaviour of laminated anisotropic plates.

2 CONSTITUTIVE RELATIONS AND POTENTIAL ENERGY

Consider a rectangular laminated anisotropic plate which is composed of fibers reinforced layers. The laminated anisotropic plate with a total thickness (t) and each layer or ply has its engineering constants in its local coordinate axes making an angle θ with the positive *x-axis* in the global coordinate system as shown in Figure 1. In the sense of CLPT, the laminated anisotropic plate is modeled as an equivalent single layer (ESL) oriented at the mid-surface. The stiffness rigidities components of the resultant ESL are evaluated by integrating the reduced stiffness matrix [Q] over the plate's thickness which depends on the material's elastic properties and defined in different standards textbooks [3], [4], [15], [33]. The reference datum plane is the mid-surface plane of the plate at $z = 0$ and the location of the top and bottom surfaces of the k^{th} layer is defined by h_{k-1} and h_k from the reference surface, respectively. Consequently, the thickness (t_k) of the k^{th} ply is evaluated by subtracting the location of the surface (h_{k-1}) from the location of the bottom one (h_k), where $k = 1,2,3, \ldots, n$. The components of the stiffness matrices are influenced by the orientation and stacking sequence of the individual layer and can be evaluated as follows:

$$A_{ij} = \sum_{k=1}^{n} \left[\bar{Q}_{ij}\right]_k (h_k - h_{k-1}) \tag{1.a}$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left[\bar{Q}_{ij}\right]_k \left(h_k^2 - h_{k-1}^2\right) \tag{1.b}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left[\bar{Q}_{ij}\right]_k \left(h_k^3 - h_{k-1}^3\right) \tag{1.c}$$

where $[\bar{Q}_{ij}]_k$ is the transformed components of the reduced stiffness matrix [Q] of the k^{th} layer.

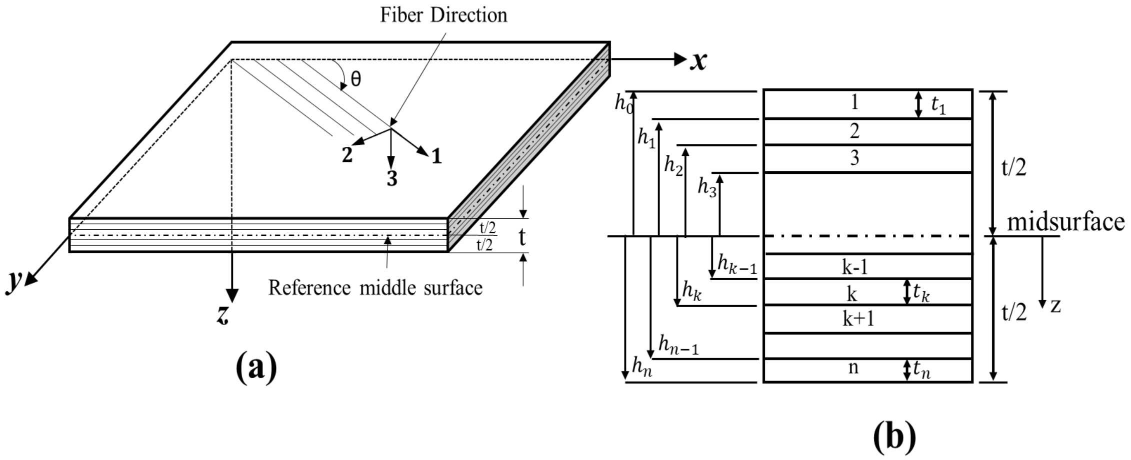

Figure 1: (a) Laminated anisotropic plate; (b) Locations of plies in the laminated plate

The strains at any point of the laminated anisotropic plate can be expressed in terms of displacements (u, v, and w) as follows:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} - z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} u,_x \\ v,_y \\ u,_y + v,_x \end{Bmatrix} - z \begin{Bmatrix} w,_{xx} \\ w,_{yy} \\ 2w,_{xy} \end{Bmatrix} \tag{2}$$

where the subscript proceeded by a comma represents differentiation; $\varepsilon_{x0}, \varepsilon_{y0}$ and γ_{xy0} represent the strains of a point at the mid-surface due to membrane effect; and κ_x, κ_y and κ_{xy} represent the strains due to curvature at a point across the thickness.

The resultant in-plane forces (N_x, N_y, N_{xy}) and bending moments (M_x, M_y, M_{xy})are expressed as in Eq. (3).

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} u,_x \\ v,_y \\ u,_y + v,_x \\ w,_{xx} \\ w,_{yy} \\ 2w,_{xy} \end{Bmatrix} \tag{3}$$

The potential energy (Π) of the anisotropic composite plate under lateral load is expressed by:

$$\Pi = U - W \tag{4}$$

where U is the total strain energy stored in the plate (Eq. 5) and W is the work done by the applied load (Eq. 6).

$$U = \frac{1}{2} \iint_A \left(N_x\, \varepsilon_{x0} + N_y\, \varepsilon_{y0} + N_{xy}\, \gamma_{xy0} + M_x\, \kappa_x + M_y\, \kappa_y + M_{xy}\, \kappa_{xy} \right) dA \tag{5}$$

$$\mathrm{W} = \iint_A q\, w\, dA \tag{6}$$

Substitution of the involved terms from Eqs. (2 and 3) into Eq. (5), and rearrangement yields the total strain energy as follows:

$$U = U^m + U^c + U^b \tag{7}$$

where U^m, U^c, and U^b represent the strain energy due to pure membrane effect (Eq. (8)), coupling effect (Eq. (9)), and bending effect (Eq. (10)), respectively,

$$U^m = \frac{1}{2}\iint_A \Big(A_{11}\, u_{,x} + A_{22}\, v_{,y} + 2A_{12}\, u_{,x}\, v_{,y} + A_{66}(u_{,y} + v_{,x})^2 \quad (8)$$
$$+ 2A_{16}\, u_{,x}\big(u_{,y} + v_{,x}\big) + 2A_{26}\, v_{,y}\big(u_{,y} + v_{,x}\big)\Big)\, dA$$

$$U^b = \frac{1}{2}\iint_A \big(D_{11} w_{,xx}^2 + D_{22} w_{,yy}^2 + 2\, D_{12}\, w_{,xx} w_{,yy} + 4D_{66} w_{,xy}^2 + 4\, D_{16}\, w_{,xx} w_{,xy} \quad (9)$$
$$+ 4\, D_{26}\, w_{,yy} w_{,xy}\big) dA$$

$$U^c = -\iint_A \Big(B_{11}\, u_{,x}\, w_{,xx} + B_{22}\, v_{,y}\, w_{,yy} + B_{12}\big(\, u_{,x}\, w_{,yy} + v_{,y}\, w_{,xx}\big)$$
$$+ 2\, B_{66}\, w_{,xy}(u_{,y} + v_{,x}) + B_{16}\Big(2u_{,x}\, w_{,xy} + w_{,xx}\big(u_{,y} + v_{,x}\big)\Big) \quad (10)$$
$$+ B_{26}\Big(2v_{,y}\, w_{,xy} + w_{,yy}\big(u_{,y} + v_{,x}\big)\Big)\Big)\, dA$$

3 FORMULATION OF RITZ METHOD IN A MATRIX-FORM

An approximate solution can be revealed for a laterally loaded laminated anisotropic plate bending problem utilizing Ritz method. The formulation of Ritz method based on the principle of minimum potential energy starts by selecting shape trial functions (Eq. (11)) to approximate the solutions for u, v, and w. The trial shape functions ϕ_i, χ_j and ψ_k are required to satisfy only the essential (geometric) boundary conditions, i.e. deflections and/or slopes.

$$\left.\begin{aligned} u(x,y) &= C_i^u \phi_i(x,y), & i &= 1, N_u \\ v(x,y) &= C_j^v \chi_j(x,y), & j &= 1, N_v \\ w(x,y) &= C_k^w \psi_k(x,y), & k &= 1, N_w \end{aligned}\right\} \quad (11)$$

where N_u, N_v, and N_w are the number of terms for each of the three trial functions ϕ_i, χ_j and ψ_k, respectively; and C_i^u, C_j^v and C_k^w are the corresponding unknown coefficients to be determined based on the principle of minimal potential energy.

The indicial notation is utilized to perform the required derivations of Ritz method starting by substitution of the approximated trial functions (Eq. (11)) in Eqs. (8-10). Then, carrying out the differentiation of Eq. (4) with respect to each of the unknown coefficient yields:

$$\left.\begin{aligned} \Pi_{,C_r^u} &= \mathrm{U}^m_{,C_r^u} + \mathrm{U}^c_{,C_r^u} + \mathrm{U}^b_{,C_r^u} - W_{,C_r^u} = 0, & r &= 1, N_u \\ \Pi_{,C_s^v} &= \mathrm{U}^m_{,C_s^v} + \mathrm{U}^c_{,C_s^v} + \mathrm{U}^b_{,C_s^v} - W_{,C_s^v} = 0, & s &= 1, N_v \\ \Pi_{,C_t^w} &= U^m_{,C_t^w} + U^c_{,C_t^w} + U^b_{,C_t^w} - W_{,C_t^w} = 0, & t &= 1, N_w \end{aligned}\right\} \quad (12)$$

Caring out the lengthy derivations of Eq. (12) as presented earlier by the authors [29] for isotropic plates with modification to consider only the linear behavior and to account for the coupling effect in this case of a laminated anisotropic plate. This produces zero energy differential terms as well as linear functions of the unknown constants, yielding the following linear system of equations:

$$\left.\begin{aligned} &\mathrm{U}^m_{,C_r^u} + \mathrm{U}^c_{,C_r^u} = 0, & r &= 1, N_u \\ &\mathrm{U}^m_{,C_s^v} + \mathrm{U}^c_{,C_s^v}{}_L = 0, & s &= 1, N_v \\ &\mathrm{U}^b_{,C_t^w} + \mathrm{U}^c_{,C_t^w} = W_{,C_t^w}, & t &= 1, N_w \end{aligned}\right\} \quad (13)$$

Carrying out the lengthy differentiations of Eqs. (13) results the final matrix form formulation of Ritz method, i.e.:

$$\begin{bmatrix} K_{11}^{ri} & K_{12}^{rj} & K_{13}^{rk} \\ K_{21}^{si} & K_{22}^{sj} & K_{23}^{sk} \\ K_{31}^{ti} & K_{32}^{tj} & K_{33}^{tk} \end{bmatrix} \begin{bmatrix} C_i^u \\ C_j^v \\ C_k^w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \iint_A q\, \psi_t \, dA \end{bmatrix} \tag{14}$$

where the components of the developed matrix form in (Eq. (14)) are presented by the expressions listed in Eqs.(15a-i) and the indices are according to:

$$r, i = 1, N_u \qquad s, j = 1, N_v \quad \text{and} \qquad k, l, m, t = 1, N_w$$

$$K_{11}^{ri} = \iint_A \left(A_{11} \phi_{i,x}\, \phi_{r,x} + A_{66}\, \phi_{i,y}\, \phi_{r,y} + A_{16} \left[\phi_{i,x}\, \phi_{r,y} + \phi_{i,y}\, \phi_{r,x}\right]\right) dA \tag{15.a}$$

$$K_{12}^{rj} = \iint_A \left(A_{12}\, \chi_{j,y}\, \phi_{r,x} + A_{66}\, \chi_{j,x}\, \phi_{r,y} + A_{16}\, \chi_{j,x}\, \phi_{r,x} + A_{26}\, \chi_{j,y}\, \phi_{r,y}\right) dA \tag{15.b}$$

$$\begin{aligned} K_{13}^{rk} = -\iint_A \Big(& B_{11}\, \phi_{r,x}\, \psi_{k,xx} + 2\, B_{12}\, \phi_{r,x}\, \psi_{k,yy} + 2\, B_{66}\, \phi_{r,y}\, \psi_{k,xy} \\ & + B_{16} \left[\phi_{r,y}\, \psi_{k,xx} + 2\phi_{r,x}\, \psi_{k,xy}\right] + B_{26}\, \phi_{r,y}\, \psi_{k,yy}\Big) dA \end{aligned} \tag{15.c}$$

$$K_{21}^{si} = \iint_A \left(A_{12}\, \phi_{i,x}\, \chi_{s,y} + A_{66}\, \phi_{i,y}\, \chi_{s,x} + A_{16}\, \phi_{i,x}\, \chi_{s,x} + A_{26}\, \phi_{i,y}\, \chi_{s,y}\right) dA \tag{15.d}$$

$$K_{22}^{sj} = \iint_A \left(A_{22}\, \chi_{j,y}\, \chi_{s,y} + A_{66}\, \chi_{j,x}\, \chi_{s,x} + A_{26} \left[\chi_{j,x}\, \chi_{s,y} + \chi_{j,y}\, \chi_{s,x}\right]\right) dA \tag{15.e}$$

$$\begin{aligned} K_{23}^{sk} = -\iint_A \Big(& B_{22}\, \chi_{s,y}\, \psi_{k,yy} + 2B_{12}\, \chi_{s,y}\, \psi_{k,xx} + 2B_{66}\, \chi_{s,x}\, \psi_{k,xy} + B_{16}\, \chi_{s,x}\, \psi_{k,xx} \\ & + B_{26} \left[\chi_{s,x}\, \psi_{k,yy} + 2\chi_{s,y}\, \psi_{k,xy}\right]\Big) dA \end{aligned} \tag{15.f}$$

$$\begin{aligned} K_{31}^{tk} = -\iint_A \Big(& B_{11}\, \phi_{k,x}\, \psi_{t,xx} + 2\, B_{12}\, \phi_{k,x}\, \psi_{t,yy} + 2\, B_{66}\, \phi_{k,y}\, \psi_{t,xy} \\ & + B_{16} \left[\phi_{k,y}\, \psi_{t,xx} + 2\phi_{k,x}\, \psi_{t,xy}\right] + B_{26}\, \phi_{k,y}\, \psi_{t,yy}\Big) dA \end{aligned} \tag{15.g}$$

$$\begin{aligned} K_{32}^{tk} = -\iint_A \Big(& B_{22}\, \chi_{k,y}\, \psi_{t,yy} + 2B_{12}\, \chi_{k,y}\, \psi_{t,xx} + 2B_{66}\, \chi_{k,x}\, \psi_{t,xy} + B_{16}\, \chi_{k,x}\, \psi_{t,xx} \\ & + B_{26} \left[\chi_{k,x}\, \psi_{t,yy} + 2\chi_{k,y}\, \psi_{t,xy}\right]\Big) dA \end{aligned} \tag{15.h}$$

$$\begin{aligned} K_{33}^{tk} = \iint_A \Big(& D_{11}\, \psi_{k,xx}\, \psi_{t,xx} + D_{22}\, \psi_{k,yy}\, \psi_{t,yy} + D_{12} \left[\psi_{t,xx}\, \psi_{k,yy} + \psi_{k,xx}\, \psi_{t,yy}\right] \\ & + 4\, D_{66}\, \psi_{k,xy}\, \psi_{t,xy} + 2\, D_{16} \left[\psi_{k,xx}\, \psi_{t,xy} + \psi_{t,xx}\, \psi_{k,xy}\right] \\ & + 2\, D_{26} \left[\psi_{k,yy}\, \psi_{t,xy} + \psi_{t,yy}\, \psi_{k,xy}\right]\Big) dA \end{aligned} \tag{15.i}$$

The proposed formulation (Eq. (14)) has a total number of $(N_u + N_v + N_w)$ linear equations containing the same number of unknown constants. The computations of all integrals in Eq.(15) and the solution for the unknown constants C^u, C^v and C^w of Eq. (14) can be carried out using any suitable software such Wolfram Mathematica or Maple [34], [35]. The plate problem is completely solved as the unknown coefficients are obtained, then the displacements u, v, and w are obtained in functional forms (Eq. (11)). Then, the secondary variables of the plate such as bending moments, membrane forces and stresses can be found using direct differentiation of the obtained displacements.

4 NUMERICAL EXAMPLES

The capability of the proposed formulation is verified through two numerical examples having high degree of anisotropy and free boundary conditions and confirmed by FEM results. The FEM analysis is performed using ABAQUS [36] by employing the shell element STRI3 which is on the basis of the classical (Kirchhoff) shell. The mesh is made extremely fine satisfying the convergence of the laminated anisotropic plate bending problem.

3.1 Materials, geometry and trial functions

Consider a square laminated plate with unsymmetrical layers oriented at (45°/-75°) and the plate is centered at the origin (*x=0, y=0*) and having the dimensions of *2a* × *2b* and thickness *t* as shown in Figure 2. The ply elastic material properties are: $E_1 = 206.85\,GPa$, $E_2 = 7.5845\,GPa$, $G_{12} = 4.8265\,GPa, v_{12} = 0.3$. This stacking sequence is selected to obtain a highly anisotropy wehre all components of the stiffness matrices [A], [B] and [D] are present. The geometrical properties of the laminated plate are $a = b = 5,$ and $a/t = 50$. The plate is subjected to a uniform lateral load and solved for two opposite free edges (F) and two types of boundary conditions for the other edges: once with simply supported (S) and once with clamped (C).

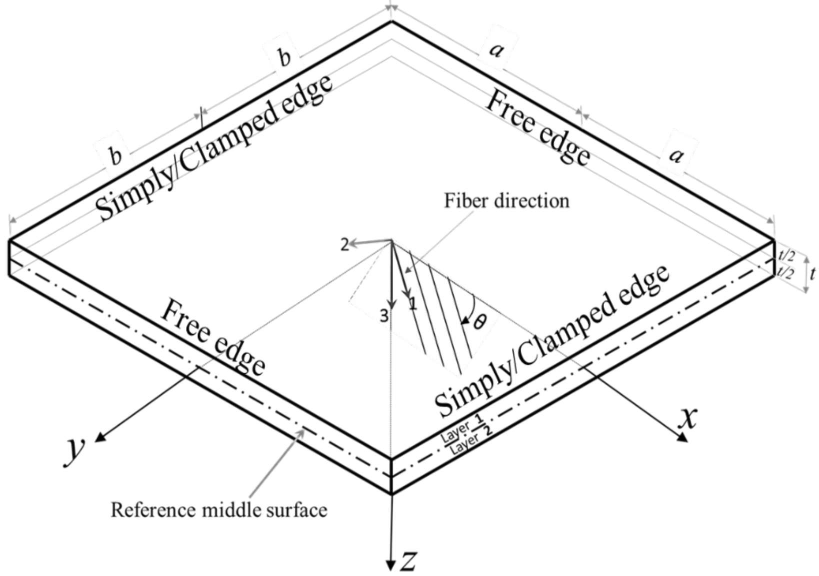

Figure 2: Laminated anisotropic plate (two layers angle-ply)

The following general polynomials satisfy the geometric boundary conditions (deflection/slope) as per Ritz method requirements:

$$
\begin{aligned}
u &= \sum_{j=0}^{n}\sum_{i=0}^{m} C_{ij}^{u}(a^2 - x^2)^{h_o}(b^2 - y^2)^{h_o}x^i y^j \\
v &= \sum_{j=0}^{n}\sum_{i=0}^{m} C_{ij}^{v}(a^2 - x^2)^{h_o}(b^2 - y^2)^{h_o}x^i y^j \qquad (16) \\
w &= \sum_{j=0}^{n}\sum_{i=0}^{m} C_{ij}^{w}(a^2 - x^2)^{h}(b^2 - y^2)^{h}x^i y^j
\end{aligned}
$$

where h is 0 for free edges, 1 for simply supported edges (preventing the lateral deflection) and 2 for clamped edges (preventing the lateral deflection and slope); and h_o is 0 for free edges and 1 for simply supported and clamped edges (the in-plane movement is prevented). The polynomial terms are truncated at maximum values of *n* and *m* as required to achieve convergence, where in both example the value of 6 is used for both *n* and *m*. The results of the numerical examples are discussed in the following section.

3.2 Results and discussions

The comparison of normalized deflection (w/t) and bending moments $\bar{M}_x = M_x(2a)^2/(t^3E_2)$ at the center and the middle of free edges of the plates is obtained using the proposed method and compared with FEM results as shown in Table 1. It is clearly shown that the proposed method predicts accurately the bending behavior of laminated anisotropic plate involving free edges. Also, the moment at the middle of the clamped edge shows good accuracy of the proposed solution. The deflection along the lines (y = 0 and y = b) is presented graphically in Figure 3 and 4 and the normalized moment $\bar{M}_x$ along the center line (y = 0) is plotted in Figure 5, which are perfectly matching the results of the FEM solution. All obtained results are for the normalized load ($q \frac{(2\,a)^4}{t\,D11} = 1$).

Table 1: comparison of the present solution with FEM for load $\left(q \frac{(2\,a)^4}{t\,D11} = 1\right)$

		SFSF plate		CFCF plate	
Point	Method	w/t	$\bar{M}_x$	w/t	$\bar{M}_x$
X =0, y=0	Present	0.0224597	0.0057501	0.0058564	0.0016267
	FEM	0.0224597	0.0057787	0.00586288	0.0016420
X =0, y=b	Present	0.0258097	0.0036250	0.0063940	0.00134171
	FEM	0.025988	0.0036323	0.0064043	0.00133784
X =a, y=0	Present	--	--	--	-0.0032860
	FEM	--	--	--	-0.0031259

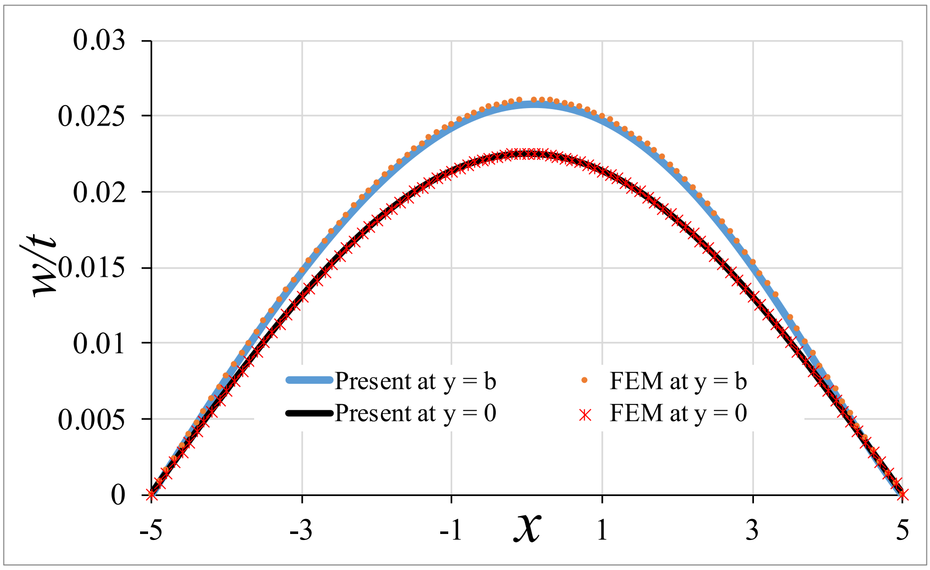

Figure 3: Deflection of SFSF plate along the free edge (y = b) and center line (y =0)

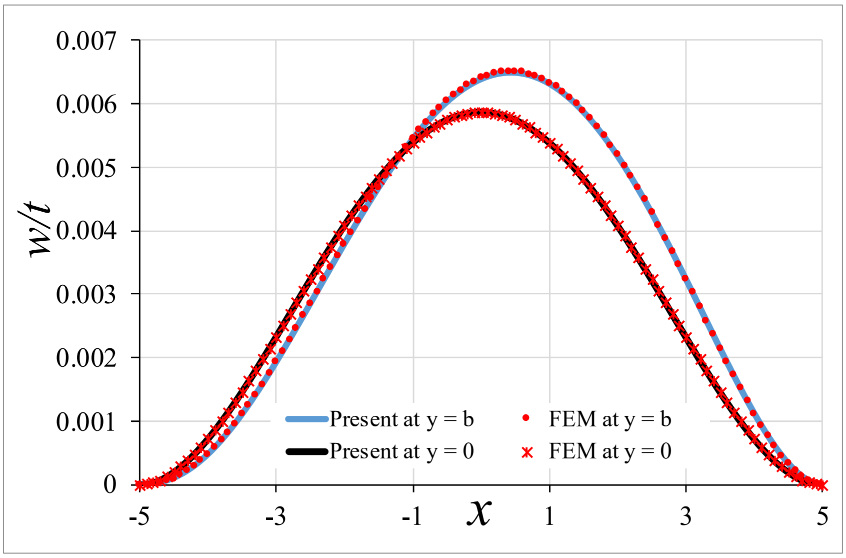

Figure 4: Deflection of CFCF plate along the free edge (y = b) and center line (y =0)

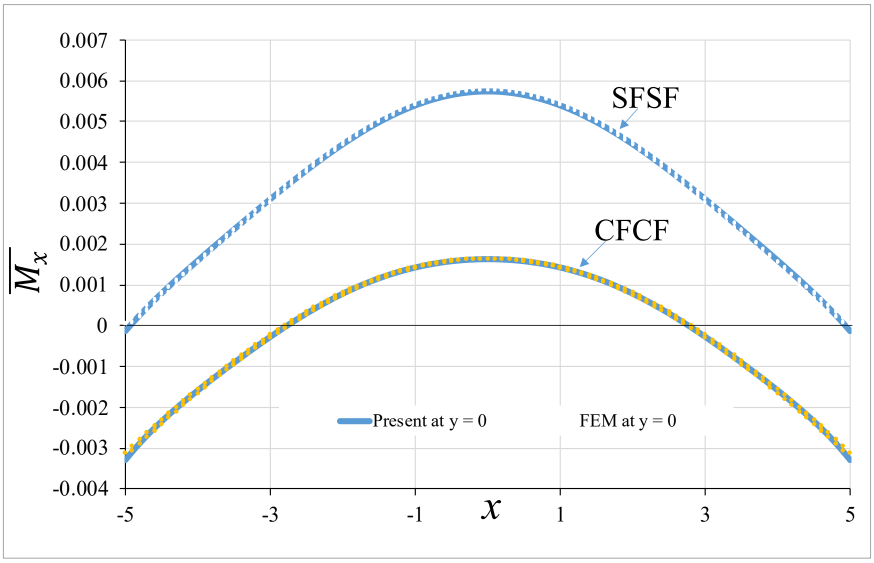

Figure 5: Normalized moment $\overline{M}_x$ of SFSF and CFCF plates along the center line (y =0)

9 CONCLUSIONS

Ritz method has been cast in a matrix form for bending behavior of a rectangular laminated anisotropic plate involving mixed boundary conditions. The formulation of Ritz method in a matrix form facilitates and automates the solution process. It also allows the accommodation of as many polynomial terms as required for the solution convergence and accuracy. Another important advantage of the proposed formulation is getting the solution in a functional form which can be easily used for further parametric and optimization design of laminated anisotropic plates. Two numerical examples involving free edges have been solved to examine the accuracy and the versatility of the proposed method. The comparison with FEM results confirm the capability of the automated Ritz method solution.

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