VARIATIONAL ANALYSIS OF LAMINATES WITH MATRIX CRACKS AND DELAMINATIONS

Vladimir Vinogradov

[1] School of Engineering, Newcastle University, Newcastle upon Tyne, NE1 7RU, UK
vladimir.vinogradov@newcastle.ac.uk

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ABSTRACT

An approximate stress field in laminates with parallel intralaminar cracks and local delamination is derived based on the principle of minimum complementary energy. Simple matrix expressions are obtained that define the effective compliance matrix, thermal expansions and curvatures, and specific heat of a cracked laminate. The method allows to analyze laminates with various stacking sequences and cracking patterns at negligible computational costs. Results for the stress distribution perturbations and changes in the effective laminate properties due to delamination cracks are presented for symmetric and non-symmetric crack patterns for varying delamination lengths.

1 INTRODUCTION

Delamination cracks in fiber reinforced composite laminates can form at various stages of the laminate service and often present the main reason for final failure. Whether the delamination cracks are formed due to monotonous, cyclic or thermal loadings, it is desirable to know their effect on the mechanical properties of the cracked laminate and the stress distribution perturbation due to the cracks.

The variational approach adopted in this paper was originated by Hashin [1] for a symmetric cross-ply $[0_n/90_m]_S$ is one of the most accurate methods to analyze the stress fields in a cracked laminate. The main idea of the approach consists of developing a stress state that satisfies all equilibrium, boundary and traction continuity conditions, and minimizes the complimentary energy of the laminate in attempt to have the best approximation for the stresses. It has been widely used and extended to different cases. One can refer to work by Nairn, who has extensively used the variational analysis, extended it to thermal loading, and used it with an energy based fracture criterion to describe crack accumulation in cross-ply laminates (see, e.g. [2] and references therein). Li and Hafeez [3] looked at a general symmetric cross-ply with symmetric periodic arrangement of transverse cracks. Vinogradov and Hashin [4] looked at an angle-ply laminate with cracks in the middle ply. Recently, the method has gained more attention, e.g. [5,6]. Vinogradov [7] expanded the approach to estimate the effective thermoelastic properties of generic laminates with parallel but not necessarily coplanar matrix cracks. Fikry et. al. [8] developed a variational analysis to explain an experimentally observed crack pattern in unidirectional laminate with a resin pocket.

The variational analysis has also been applied to study effects of delamination cracks [9,10] on the effective properties of cross-ply laminates and associated energy release rate. The present work extends the approach to generic laminates and in-plane loadings.

2 ADMISSIBLE STRESS FIELD

Consider an n-ply laminate sample in the xy plane under a uniform in-plane membrane forces N_x, N_y, N_{xy} and moments M_x, M_y, M_{xy}. We next impose an arbitrary state of damage in a family of plies defined by a certain fibre orientation θ^*, when the intralaminar cracks are parallel to the fiber direction, but are not necessarily coplanar. One can then rotate the coordinate system such that the crack surfaces are all normal to the x-axis and the cracked plies become 90° plies (see Fig. 1). The front of interlaminar cracks are assumed to be parallel to axis y.

For an arbitrary orientation of plies in the laminate, when there are no cracks, the in-plane stresses $\sigma_{xx}^{0(m)}$, $\sigma_{yy}^{0(m)}$ and $\sigma_{xy}^{0(m)}$ in any ply (m) are linear function of the transverse coordinate z, which can be obtained from a simple analysis using the classical laminate theory, and the rest of the stress tensor components σ_{yz}^{0}, σ_{xz}^{0} and σ_{zz}^{0} are equal to zero. The cracks introduce stress perturbations, which are denoted $\sigma_{ij}^{(m)}$.

Let us represent the stresses in the m-th ply of the cracked material as a superposition of the stresses in the uncracked material and perturbation stresses due to the presence of the cracks

$$\sigma_i^{C(m)}(x,z) = \sigma_i^{0(m)}(z) - \sigma_i^{(m)}(x,z). \tag{1}$$

It is assumed that the in-plane perturbation stresses $\sigma_{xx}^{(m)}$, $\sigma_{yy}^{(m)}$ and $\sigma_{xy}^{(m)}$ in ply m are linear functions of z everywhere in the cracked laminate, i.e. for every coordinate x. According to this assumption the in-plane perturbation stresses can be expressed in terms of yet unknown functions $\phi_i^{(m)}(\xi)$ and $\psi_i^{(m)}(\xi)$:

$$\sigma_i^{(m)}(\xi,z) = \phi_i^{(m)}(\xi) + \psi_i^{(m)}(\xi)\,\zeta, \qquad i = (1 = xx, 2 = yy, 6 = xy), \tag{2}$$

where $\xi = \frac{x}{t_0}$ is the dimensionless x-coordinate, t_0 is an arbitrary normalization thickness, which is chosen as the thickness of a lamina. ζ is the dimensionless z-coordinate, defined as

$$\zeta = (z - \bar{z}_m)/t_m \tag{3}$$

and varying between -1/2 to 1/2 within each ply, z_m is the z-coordinate of the middle of ply m, t_m is the thickness of the ply m (Figrue 1).

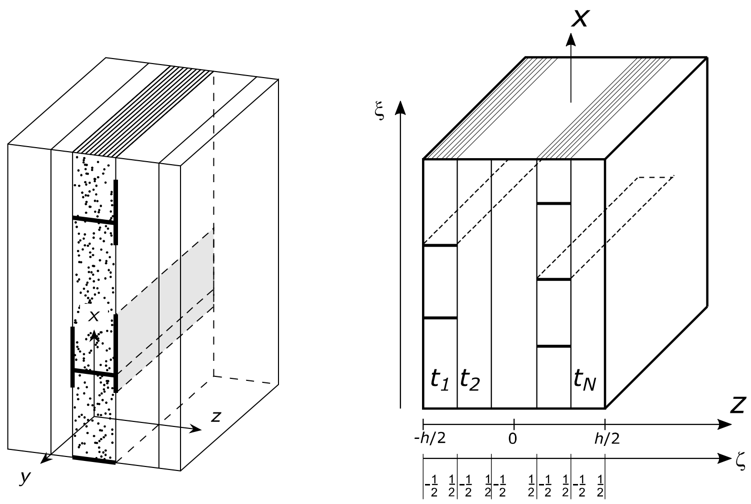

Figure 1. Schematic presentation of a cracked laminate with matrix and delamination cracks.

The in–plane stresses (2) can be substituted into the differential equilibrium equations to get the out-of-plane stresses $\sigma_{xz}, \sigma_{yz}, \sigma_{zz}$. The resultant stresses satisfy zero traction at $z = -h/2$ and traction continuity at the interfaces between plies, and can explicitly be expressed in terms of the functions $\phi(\xi)$, $\psi(\xi)$ and their first and second derivatives:

$$\boldsymbol{\sigma}^{(m)} = \mathbf{A}_0^{(m)}(\zeta)\,\mathbf{f}(\xi) + \mathbf{A}_1^{(m)}(\zeta)\,\mathbf{f}'(\xi) + \mathbf{A}_2^{(m)}(\zeta)\,\mathbf{f}''(\xi), \tag{4}$$

where $\boldsymbol{\sigma}(\xi,\zeta) = \left(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}\right)^T$ is a (6×1) vector of the stress tensor components, $\mathbf{f}(\xi)$ is a $(6N \times 1)$ vector of unknown functions:

$$\mathbf{f}(\xi) = \left(\phi_1^{(1)}, \phi_2^{(1)}, \phi_6^{(1)}, \psi_1^{(1)}, \psi_2^{(1)}, \psi_6^{(1)}, \phi_1^{(2)}, \dots\right)^T, \tag{5}$$

and the matrices $\mathbf{A}_0^{(m)}$, $\mathbf{A}_1^{(m)}$, $\mathbf{A}_2^{(m)}$ are the ζ-dependant coefficient, whose elements are explicitly given in [7].

Independently of the membrane loading applied to the laminate, the total forces and moments formed by the perturbation stresses should vanish. This implies that for perturbation forces

$$t_0 \sum_{m=1}^{n} \lambda_m \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_i^{(m)}(\xi,\zeta)\, d\zeta = 0, \quad i = (1 \equiv xx, 2 \equiv yy, 6 \equiv xy) \tag{6}$$

and for perturbation moments

$$t_0^2 \sum_{m=1}^{n} \lambda_m^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_i^{(m)}(\xi,\zeta)\, \zeta\, d\zeta = 0, \quad i = (1 \equiv xx, 2 \equiv yy, 6 \equiv xy) \tag{7}$$

Eq. (6) and (7) form six constraints that must hold for any coordinate x. It is follows that these constraints are sufficient to satisfy zero traction condition at the external surface $z = h/2$ [7].

In the intervals where delamination cracks are present, the zero-traction condition must hold for the delaminated surfaces, which is ensured if the axial force $N_{xx}^{(d)}$and shear force $N_{xy}^{(d)}$ in the delaminated sub-laminate remain constant along the delaminated region, while the bending moment $M_{xx}^{(d)}$ is a linear function of ξ

$$N_{xx}^{(d)} = t_0 \sum_{m=1}^{n_d} \lambda_m \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_{xx}^{(m)}(x,\zeta)\, d\zeta = c_1, \qquad N_{xy}^{(d)} = t_0 \sum_{m=1}^{n_d} \lambda_m \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_{xy}^{(m)}(\xi,\zeta)\, d\zeta = c_2 \tag{8}$$

$$M_{xx}^{(d)} = t_0^2 \sum_{m=1}^{n_d} \lambda_m^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_{xx}^{(m)}(\xi,\zeta)\, \zeta\, d\zeta = c_3 + c_4\xi, \tag{9}$$

The six constraints of Eqs. (6) and (7) can be written in the matrix form

$$\mathbf{B}_{\mathrm{eq}}\mathbf{f} = 0, \tag{10}$$

where $\mathbf{B}$ is a $(6 \times 6N)$ matrix. Constraints (8) and (9) are written as

$$\mathbf{B}_{\mathrm{d}}\, \mathbf{f} = \mathbf{c}_{\mathrm{d}}(\xi), \tag{11}$$

where $\mathbf{B}_d$ is a $(3 \times 6N)$ matrix and $\mathbf{c}_{\mathrm{d}}(\xi)$ represents the combination of right-hand sides of Eqns. (8)-(9). Additional constraints as in Eq. (11) must be appended if the interval contains several delaminated interfaces. Eqs. (10) and (11) are now combined

$$\mathbf{B}\, \mathbf{f}(\xi) = \mathbf{c}(\xi). \tag{12}$$

These constraints must be satisfied at each location ξ along a delamination crack and must be included in the minimization of the complementary energy.

3 VARIATIONAL SOLUTION

According to the variational approach, employed in the present paper, the unknown functions $\mathbf{f}(\xi)$ are determined using the principle of minimum complementary energy. The complementary energy in terms of the perturbation stresses is given as:

$$U_C = U_C^0 + \frac{1}{2}\int_V \boldsymbol{\sigma}^T \mathbf{S}\,\boldsymbol{\sigma}\, dV = U_C^0 + \frac{t_0^2}{2}\int_L \sum_{p,q=0,1,2} \mathbf{f}^{(p)^T} \mathbf{M}_{pq} \mathbf{f}^{(q)} d\xi\,, \tag{13}$$

where U_C^0 is the complementary energy of the uncracked laminate, $\mathbf{S}(\mathbf{x})$ is the local compliance tensor, $\mathbf{f}^{(p)}$ denotes the p-th derivative of $\mathbf{f}(\xi)$, and

$$\mathbf{M}_{pq} = \sum_{m=1}^{n} \lambda_m \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{A}_p^{(m)^T}(\zeta)\,\mathbf{S}^{(m)} \mathbf{A}_q^{(m)}(\zeta)\, d\zeta \tag{14}$$

Elements of the integrand matrix in (14) represent polynomials of ζ of order up to six and can be calculated numerically to any degree of accuracy. Explicit expressions for elements of matrices $\mathbf{M}_{pq}$ can be found in [7].

Minimization of the complementary energy reduces to the problem of determining functions $\mathbf{f}$ that minimize the integral in Eq. (13)

$$I = \min_{\mathbf{f}(\xi)} \int_L F(\xi) d\xi, \tag{15}$$

with the integrand having the following form:

$$F(\xi) = \mathbf{f}^T \mathbf{M}_{00}\mathbf{f} + \mathbf{f}^T \mathbf{M}_{02}\mathbf{f}'' + \mathbf{f}''^T \mathbf{M}_{20}\mathbf{f} + \mathbf{f}' \mathbf{M}_{11}\mathbf{f}' + \mathbf{f}''^T \mathrm{M}_{22}\mathbf{f}'' \tag{16}$$

subject to constraints (12), which are added to the Lagrangian forming a new augmented functional:

$$I = \min_{\mathbf{f},\boldsymbol{\omega}} \int_L [F(\xi) + 2\boldsymbol{\omega}^T(\xi)(\mathbf{B}\,\mathbf{f}(\xi) - \mathbf{c}(\xi))] d\xi, \tag{17}$$

where $2\boldsymbol{\omega}(\xi)$ is the vector of Lagrange multipliers.

The Euler-Lagrange equations for this functional lead to the system

$$\begin{bmatrix} \mathbf{M}_0 & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\omega} \end{bmatrix} + \begin{bmatrix} \mathbf{M}_2 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{f}'' \\ \boldsymbol{\omega}'' \end{bmatrix} + \begin{bmatrix} \mathbf{M}_4 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{f}^{(iv)} \\ \boldsymbol{\omega}^{(iv)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{c}(\xi) \end{bmatrix}, \tag{18}$$

where $\mathbf{M}_0 = \mathbf{M}_{00}$, $\mathbf{M}_2 = \mathbf{M}_{02} + \mathbf{M}_{20} - \mathbf{M}_{11}$ and $\mathbf{M}_4 = \mathbf{M}_{22}$. Eliminating the Lagrange multipliers one obtains:

$$\mathbf{M}_0\mathbf{f} + \mathbf{H}\mathbf{M}_2\mathbf{f}'' + \mathbf{H}\mathbf{M}_4\mathbf{f}^{(iv)} = \mathbf{B}^T(\mathbf{B}\,\mathbf{M}_0^{-1}\mathbf{B}^T)^{-1}\mathbf{c}(\xi), \tag{19}$$

where $\mathbf{H} = \mathbf{I} - \mathbf{B}^T(\mathbf{B}\mathbf{M}_0^{-1}\mathbf{B}^T)^{-1}\mathbf{B}\mathbf{M}_0^{-1}$ is the projection matrix onto the set of solutions satisfying Eq. (12). The general solution is a combination of the homogeneous and particular solutions. The homogeneous part satisfies the equation

$$\mathbf{M}_0\mathbf{f}_h + \mathbf{H}\mathbf{M}_2\mathbf{f}_h'' + \mathbf{H}\mathbf{M}_4\mathbf{f}_h^{(iv)} = 0, \tag{20}$$

which reduces to an eigenvalue problem and is solved by the same way as without delamination cracks [7] to give

$$\mathbf{f}_h = \sum_i c_{h,i}\mathbf{v}_i \exp(r_i\xi) = \mathbf{U}_h(\xi)\mathbf{c}_h, \tag{21}$$

where r_i is the i-th eigenvalue, $\mathbf{v}_i$ is the corresponding eigenvector and $\mathbf{c}_h$ is a vector of constant coefficients.

The right-hand side of Eq. (19) is a linear function of x, hence the particular solution is

$$\mathbf{f}_p = \mathbf{M}_0^{-1}\mathbf{B}^T(\mathbf{B}\,\mathbf{M}_0^{-1}\mathbf{B}^T)^{-1}\mathbf{c}(\xi) = \mathbf{U}_p(\xi)\mathbf{c}_p, \tag{22}$$

where $\mathbf{c}_p$ is a vector of unknown constants. in case of one delamination surface, the constants follow from Eq. (7): $\mathbf{c}_p = [c_1, c_2, c_3, c_4]^T$. Finally, the solution of the Euler Lagrange equations can be written in the form

$$\mathbf{f} = \mathbf{f}_p + \mathbf{f}_h = \mathbf{U}_h(\xi)\mathbf{c}_h + \mathbf{U}_p(\xi)\mathbf{c}_p = \mathbf{U}(\xi)\mathbf{c} \tag{23}$$

where the coefficients $\mathbf{c}$ are defined using boundary conditions at the planes of transverse cracks, continuity of traction between intervals and other possible boundary conditions at the end planes of the intervals, such as periodicity or reflection symmetry.

Typically, the number of boundary conditions is insufficient to determine all the coefficients. In order to obtain them, the complementary energy is minimized considering constraints (12). Substitution (23) into (15), leads to the following quadratic form in terms of independent constants $\mathbf{c}$:

$$\min_{\mathrm{c}}\left[I = \sum_{k=1}^{n} \mathbf{c}_k^T \mathbf{W}_k(L_k)\mathbf{c}_k = \mathbf{c}^T\mathbf{W}\mathbf{c}\right], \qquad \mathrm{s.t.}\quad \mathbf{B}_{\mathrm{bc}}\,\mathbf{c} = \mathbf{b} \tag{24}$$

where the sum is over all the joined intervals of lengths L_k having different delamination pattern or separated by transverse cracks. Matrices $\mathbf{W}_k(L_k)$ have a closed form expression in terms of matrices $\mathbf{U}$ and $\mathbf{M}$ (omitted here). Using again the method of Lagrange Multipliers, the solution of the problem can be derived to be

$$\mathbf{c} = \mathbf{W}_{\mathrm{sym}}^{-1}\mathbf{B}_{\mathrm{bc}}^T\left(\mathbf{B}_{\mathrm{bc}}\mathbf{W}_{\mathrm{sym}}^{-1}\mathbf{B}_{\mathrm{bc}}^T\right)^{-1}\mathbf{b} \tag{25}$$

which leads to the final stress distribution and the complementary energy in the form

$$U_C = U_C^0 + \frac{t_0^2}{2}\mathbf{b}^T\left(\mathbf{B}_{\mathrm{bc}}\mathbf{W}_{\mathrm{sym}}^{-1}\mathbf{B}_{\mathrm{bc}}^T\right)^{-1}\mathbf{b} \tag{26}$$

The complementary elastic energy can be written in terms of its effective compliance matrix $\mathbf{ABD}^*$, thermal expansions and specific heat of the cracked laminate, and the effect of the cracks on the effective properties/engineering constants can be investigated.

4 NUMERICAL EXAMPLES

Although the derived solution for the stress distribution and effective properties of a laminate is concise and requires negligible computational efforts, it is very robust and applicable to a variety of crack geometries and laminate systems. It has also been shown [7] that the results for effective thermomechanical properties of laminates with transverse cracks (no delamination cracks) agree very well with experimental data.

In Fig. 4 the results of the axial stress is shown, formed in $[\pm 45/90_2]_S$ GFRP laminate due to axial tensile loading. The thick black solid line indicates the location of cracks. The left figure shows the stress distribution for a symmetric crack pattern, while the right figure shows the stress distribution for the antisymmetric delamination geometry. In both the cases reflection symmetry boundary conditions were applied requesting $\sigma_{xz} = 0$ at the planes of symmetry. In these examples, the plies are subdivided, and more nonlinear stress distribution along the thickness direction is revealed, with clear stress singularities at the tips of the delamination cracks.

Fig. 5 shows the axial stress distribution for the same geometry, but for the periodic Z-type arrangement of the delamination cracks. In this case the symmetry boundary conditions of the previous examples were replaced with periodic boundary conditions, resulting in quite a different stress field.

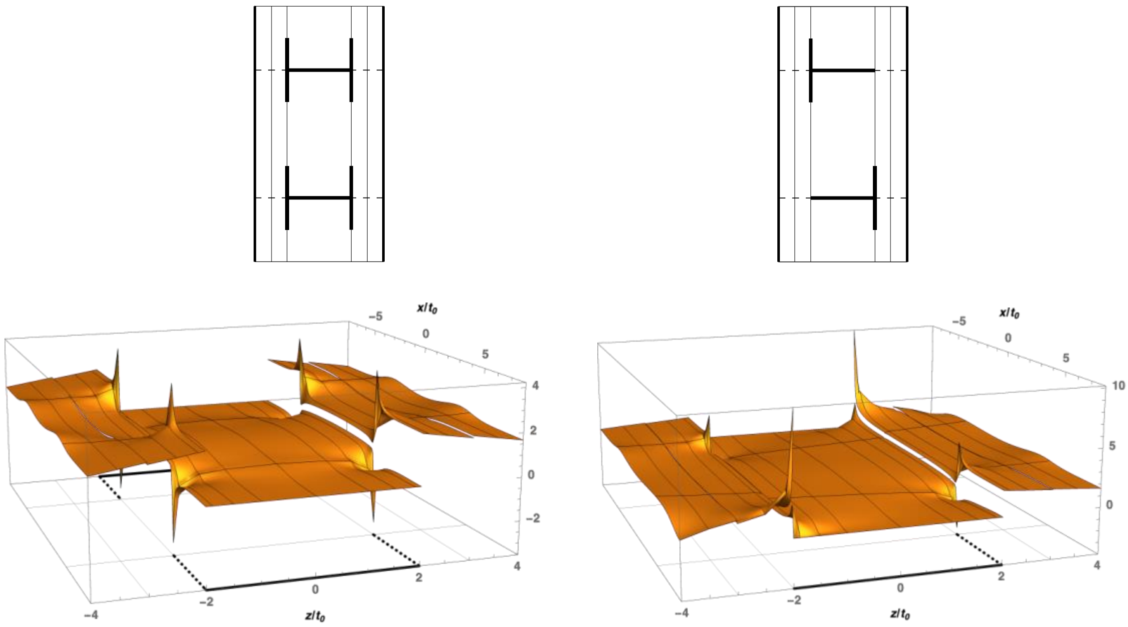

Figure 4. Stress distribution σ_{xx} in cracked [±45/90$_2$]$_S$. Left: symmetric delaminations, Right: antisymmetric delaminations

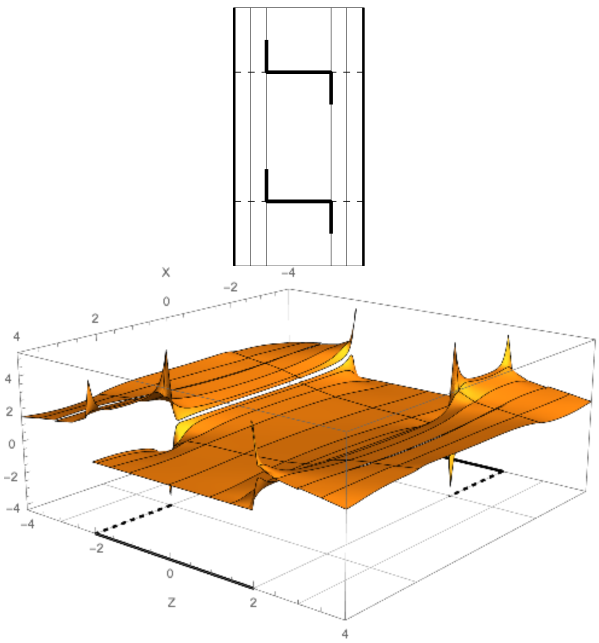

Figure 5. Stress distribution σ_{xx} in cracked [±45/90$_2$]$_S$ for Z-delamination (periodic boundary conditions)

Figs. 6 and 7 show the results for the effective axial Young's modulus and coefficient of thermal expansion of the cracked [±45/90$_2$]$_S$ as functions of delamination length. The normalized spacing between transverse cracks in 90 ply is indicated in Figs. 6 and 7 for each pair of curves, when the solid lines correspond to the symmetric pattern and the dashed lines correspond to the antisymmetric delamination pattern. It is interesting to note that due to the nature of the developed solution, it is possible to analyze laminates with very short delamination cracks, as well as cracks that extend to almost the entire length of a laminate, which is quite difficult to implement using FE without introducing a very fine mesh size.

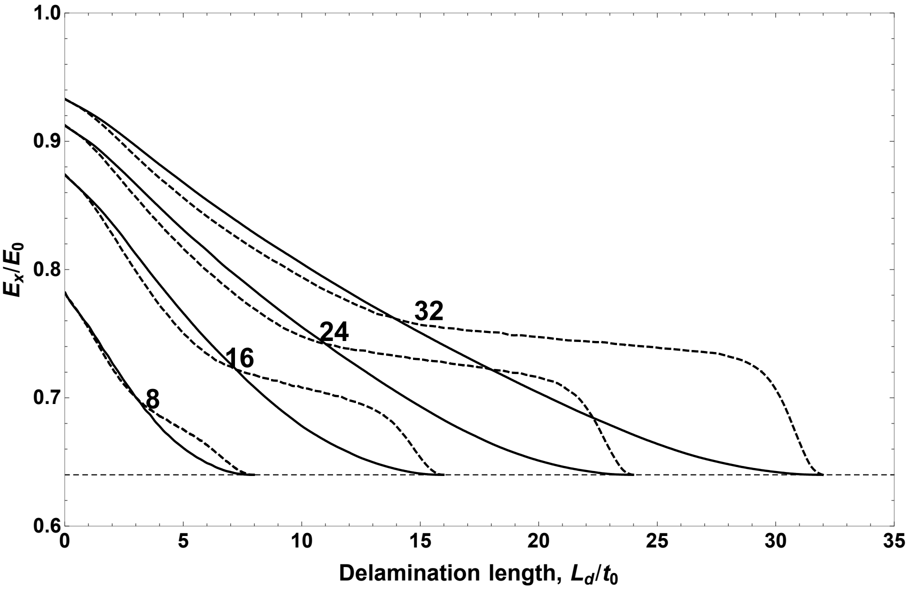

Figure 6. Effective axial Young’s modulus vs delamination length in cracked $[\pm 45/90_2]_S$ for various initial transverse crack densities

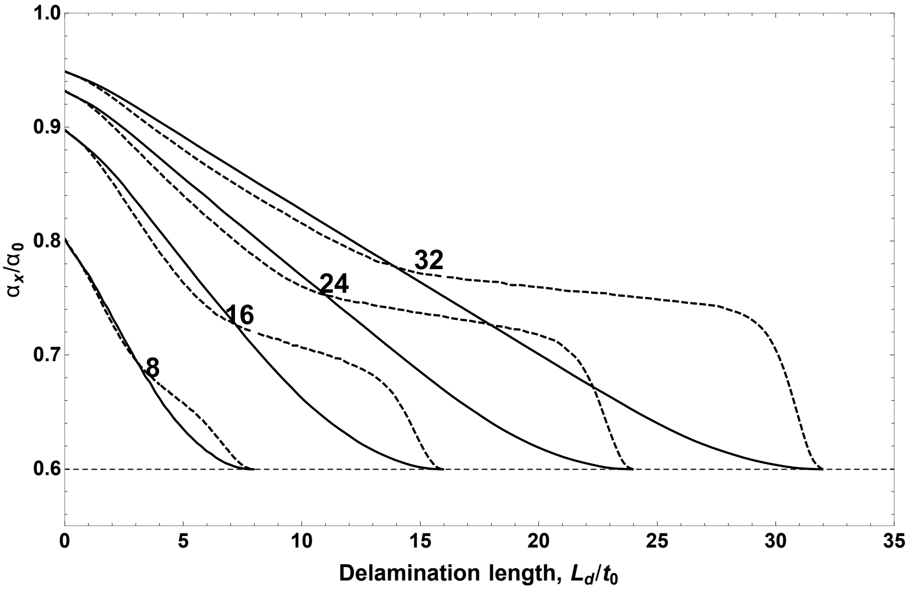

Figure 7. Effective axial thermal expansion coefficient vs delamination length in cracked $[\pm 45/90_2]_S$ for various initial transverse crack densities.

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