STUDY OF A CRACK IN A FUNCTIONALLY GRADED LAYERED STRUCTURE UNDER THERMAL LOAD

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Abstract

In this paper, the thermal fracture behavior for a crack perpendicular to the interface in a functionally graded layered structure (FGLS) with a functionally graded interfacial layer is investigated. The plane strain state is considered. During the analytical procedure, integral transform methods are used to obtain the displacement and stress expressions. Then, by introducing an auxiliary function and using the residue theory and the singular integral equation method, the thermal stress intensity factor (TSIF) is calculated. Particularly, a crack intersecting the interface is considered. Some representative structure with different nonhomogeneity material properties and geometric parameters are analyzed and the corresponding TSIF is presented. The influences of the nonhomogeneity constants and the geometry parameters on the TSIFs are analyzed.

1 Introduction

Functionally graded materials (FGMs) belong to composite materials with the material properties varying continuously from place to place according to the performance requirements. Functionally graded layered structures (FGLSs) have great potential in thermal gradient structures and metal/ceramic joining. In the FGLSs, the material properties vary from one layer to another layer in a continuous manner. Therefore, the mismatch of material properties between different layers, which may involve the high residual and thermal stresses and result in the cracking, can be eliminated. In the past decades, FGLSs, such as the functionally graded ceramic coating/metal substrate system, have absorbed great attention. In the field of fracture mechanics, most of the past investigations are concentrated on the structures with a crack located on the interface or in one functionally graded layer [1-5]. However, few investigations have been carried out on the crack intersecting the interface in the functionally graded structures.

In this paper, the thermal fracture behaviors for a crack intersecting the interface in a FGLS are analyzed. The plane strain state is considered. During the analytical procedure, integral transform methods are used to obtain the displacement and stress expressions. Then, by introducing an auxiliary function and using the residue theory and the singular integral equation method, the present problem is reduced to a singular integral equation which can be solved through numerical methods.

2 Formulation of the Problem

The functionally graded layered structure is shown in Fig. 1. Layer 1 and Layer 3 are both homogeneous layers with the thicknesses h_1 and h_3. Layer 2 is a functionally graded interfacial layer with the thickness h_2. Assume $h = h_1 + h_2 + h_3$, μ_j, λ_j and α_j ($j = 1,2,3$) are the shear moduli, the coefficients of heat conduction and the thermal expansion coefficients of three layers, respectively. The surface temperatures are T_{01} and T_{02}. The initial temperature is assumed as T_0. The material properties are continuous across the interface. First, we define the shear modulus, the coefficient of heat conduction and the thermal expansion coefficient of the functionally graded interfacial layer as

$$\mu_2(x) = \mu_{02} e^{\delta x} \tag{1}$$

$$\lambda_2(x) = \lambda_{02} e^{\eta x} \tag{2}$$

$$\alpha_2(x) = \alpha_{02} e^{\omega x} \tag{3}$$

According to the continuity of the material properties, we have the properties of the other two layers

$$\mu_1 = \mu_2(h_1) = \mu_{02} e^{\delta h_1} \quad (4)$$

$$\lambda_1 = \lambda_2(h_1) = \lambda_{02} e^{\eta h_1} \quad (5)$$

$$\alpha_1 = \alpha_2(h_1) = \alpha_{02} e^{\omega h_1} \quad (6)$$

$$\mu_3 = \mu_2(h_1 + h_2) = \mu_{02} e^{\delta(h_1+h_2)} \quad (7)$$

$$\lambda_3 = \lambda_2(h_1 + h_2) = \lambda_{02} e^{\eta(h_1+h_2)} \quad (8)$$

$$\alpha_3 = \alpha_2(h_1 + h_2) = \alpha_{02} e^{\omega(h_1+h_2)} \quad (9)$$

where μ_{02}, α_{02}, λ_{02}, δ, ω and η are material constants.

If the thermo-elastic coupling effects are negligible, the crack problem can be solved by superposition. As Erdogan and Wu [2] have done, the present crack problem can be reduced to a perturbation problem and the crack surface loads resulted from the temperature distribution will be equal and opposite to the thermal stresses in the absence of the crack. Therefore, for the structure in Fig.1, the temperature distribution and thermal stresses will be firstly obtained in the absence of the crack.

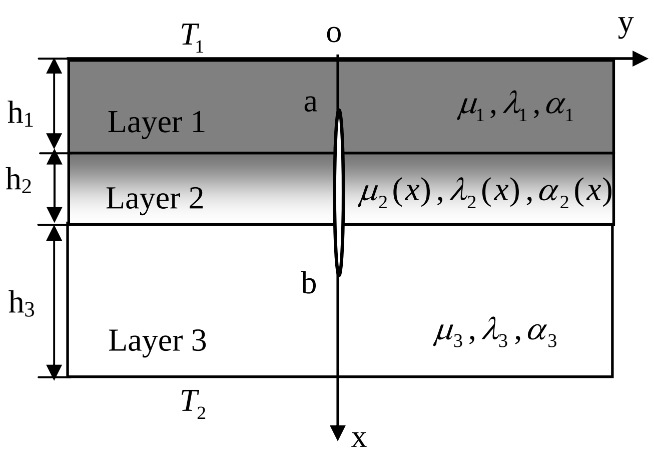

Fig. 1. Functionally graded layered structure with a crack intersecting the interface

The steady state equation can be written as

$$\partial[\lambda_j(x)\partial T_j(x)/\partial x]/\partial x = 0, \quad (j = 1, 2, 3) \quad (10)$$

Using the thermal boundary and continuity conditions,

$$T_1(x = 0) = T_{01} \quad (11)$$

$$T_3(x = h) = T_{02} \quad (12)$$

$$T_1(x = h_1) = T_2(x = h_1) \quad (13)$$

$$T_2(x = h_1 + h_2) = T_3(x = h_1 + h_2) \quad (14)$$

$$\lambda_1 \left.\frac{\partial T_1(x)}{\partial x}\right|_{x=h_1} = \lambda_2 \left.\frac{\partial T_2(x)}{\partial x}\right|_{x=h_1} \quad (15)$$

$$\lambda_2 \left.\frac{\partial T_2(x)}{\partial x}\right|_{x=h_1+h_2} = \lambda_3 \left.\frac{\partial T_3(x)}{\partial x}\right|_{x=h_1+h_2} \quad (16)$$

the temperature fields for the three layers can be obtained as

$$T_1(x) = T_{01} + \frac{(T_{02} - T_{01})}{w_1 + w_2 + w_3} \int_0^x \frac{dx}{\lambda_1(x)} \quad (17)$$

$$T_2(x) = \frac{T_{02} w_1 + T_{01}(w_2 + w_3)}{w_1 + w_2 + w_3} + \frac{(T_{02} - T_{01})}{w_1 + w_2 + w_3} \int_{h_1}^x \frac{dx}{\lambda_2(x)} \quad (18)$$

$$T_3(x) = \frac{T_{02}(w_1 + w_2) + T_{01} w_3}{w_1 + w_2 + w_3} + \frac{(T_{02} - T_{01})}{w_1 + w_2 + w_3} \int_{h_1+h_2}^x \frac{dx}{\lambda_3(x)} \quad (19)$$

where w_1, w_2 and w_3 are known constants.

Then, the thermal stress distribution in the absence of the crack can be solved. For plane strain state, we have

$$\sigma_{jyy}^T(x) = \frac{E_j(x)}{1-\nu_j^2}\{A_j x + B_j - \alpha_j(x)(1+\nu_j)[T_j(x) - T_0]\}, \quad (j = 1, 2, 3) \quad (20)$$

where, A_j and B_j $(j = 1, 2, 3)$ are coefficients which can be determined from the displacement and stress boundary and continuity conditions of the structure. If the structure is only subjected to the temperature distribution without constraining along its boundaries, the resultant force and moment in the structure are zero. Namely,

$$\begin{cases} \int_0^{h_1} \sigma_{1yy}^T(x)dx + \int_{h_1}^{h_1+h_2} \sigma_{2yy}^T(x)dx \\ +\int_{h_1+h_2}^{h} \sigma_{3yy}^T(x)dx = 0 \\ \int_0^{h_1} \sigma_{1yy}^T(x)xdx + \int_{h_1}^{h_1+h_2} \sigma_{2yy}^T(x)xdx \\ +\int_{h_1+h_2}^{h} \sigma_{3yy}^T(x)xdx = 0 \end{cases} \tag{21}$$

Together with the continuity conditions of stresses and displacements, the unknowns A_j and B_j ($j=1,2,3$) can be determined. According to the above description, in the following crack analysis, $-\sigma_{jyy}^T(x)$ ($j=1,2,3$) will be used as the crack surface traction.

In the following analysis regarding the crack, the boundary and continuity conditions can be written as

$$\sigma_{1xx}(0,y) = 0 \tag{22}$$

$$\sigma_{1xy}(0,y) = 0 \tag{23}$$

$$\sigma_{1xx}(h_1,y) = \sigma_{2xx}(h_1,y) \tag{24}$$

$$\sigma_{1xy}(h_1,y) = \sigma_{2xy}(h_1,y) \tag{25}$$

$$\sigma_{2xx}(h_1+h_2,y) = \sigma_{3xx}(h_1+h_2,y) \tag{26}$$

$$\sigma_{2xy}(h_1+h_2,y) = \sigma_{3xy}(h_1+h_2,y) \tag{27}$$

$$v_1(h_1,y) = v_2(h_1,y) \tag{28}$$

$$u_1(h_1,y) = u_2(h_1,y) \tag{29}$$

$$v_2(h_1+h_2,y) = v_3(h_1+h_2,y) \tag{30}$$

$$u_2(h_1+h_2,y) = u_3(h_1+h_2,y) \tag{31}$$

$$\sigma_{2xx}(h,y) = 0 \tag{32}$$

$$\sigma_{2xy}(h,y) = 0 \tag{33}$$

$$v_1(x,0) = v_2(x,0) = v_3(x,0) = 0, \quad x \notin (a,b) \tag{34}$$

$$\sigma_{1xy}(x,0) = \sigma_{2xy}(x,0) = 0 \tag{35}$$

$$\sigma_{jyy}(x,0) = -\sigma_{jyy}^T(x), \quad a<x<b \tag{36}$$

In the condition (36), j $(=1,2,3)$ corresponds to the layer in which *x* lies.

The constitutive equations can be written as

$$\begin{cases} \sigma_{jxx} = \dfrac{\mu_j}{k_j-1}\left[\left(1+k_j\right)\dfrac{\partial u_j}{\partial x} + \left(3-k_j\right)\dfrac{\partial v_j}{\partial y}\right] \\ \sigma_{jyy} = \dfrac{\mu_j}{k_j-1}\left[\left(1+k_j\right)\dfrac{\partial v_j}{\partial y} + \left(3-k_j\right)\dfrac{\partial u_j}{\partial x}\right] \\ \sigma_{jxy} = \mu_j\left[\dfrac{\partial u_j}{\partial y} + \dfrac{\partial v_j}{\partial x}\right] \end{cases} \tag{37}$$

in which, k_j ($j=1,2,3$) are constants related to the Poisson's ratio. According to the previous work [2], the TSIFs are relatively insensitive to the Poisson's ratio. Therefore, Poisson's ratio υ is assumed to be a constant. $k_j = 3-4\upsilon$ for plane strain problem.

Using the constitutive relation (37) and equilibrium equations, the displacement expression can be expressed as

$$\begin{cases} u_j = u_j(A_{j11}, A_{j12}, A_{j21}, A_{j22}, A_{j23}, A_{j24}, x, y) \\ v_j = v_j(A_{j11}, A_{j12}, A_{j21}, A_{j22}, A_{j23}, A_{j24}, x, y) \end{cases} \tag{38}$$

where A_{j1m} and A_{j2l} ($j=1,2,3$; $m=1,2$; $l=1,\cdots,4$) are unknowns to be decided through the known conditions. Some details about the displacement expressions can be found in Refs. [4-6]. Then using the constitutive relations, the stress expressions can be obtained.

Define the following auxiliary function

$$g(x) = \frac{\partial v_j(x,0)}{\partial x}, \quad a<x<b \tag{39}$$

The known condition (34) requires

$$g(x) = 0, \quad x \notin (a,b) \tag{40}$$

$$\int_a^b g(x)dx = 0 \tag{41}$$

Then using conditions (22)-(35), the unknowns A_{j1m} and A_{j2l} can be expressed by the auxiliary function $g(x)$.

Finally, using the crack face condition (36), we can obtain the singular integral equation

$$\frac{1}{\pi}\int_a^b\left[\frac{d_j}{x-u}+k_{j1}(u,x)+\pi h_{j2}(u,x)\right]g(u)du = -e^{\frac{1-(-1)^{j-1}}{2}\delta x}\sigma_{jyy}^T(x) \quad , (j=1,2,3) \tag{42}$$

In equation (42), j corresponds to the layer in which x lies. The detailed procedure can be found in Ref. [4-6].The singular integral equation (42) will be solved by the numerical method in Ref.[7, 8].

3 Results and Discussions

In the following analysis, plane strain state is considered. Poisson's ratio is assume to be 0.33. For convenience, the TSIF will be normalized by $k_0=\sigma_0\sqrt{a_0}$. Define the quantities $a_0=(b-a)/2$, $c_0=(b+a)/2$ and $\sigma_0=E_0\alpha_0 T_0/(1-\nu)$. Here $E_{02}=2(1+\nu)\mu_{02}$. To verify the present method, we compare the normalized TSIFs of an edge crack in a functionally graded layer with the thickness h. By letting $h_2=h_3\to 0$ and $h_1=h$, the present structure in Fig.1 can be reduced to a functionally graded strip which has been studied by Erdogan and Wu [2]. The Young's modulus $E_2(x)$ can be determined by the shear modulus (1) and thermal properties $\lambda_2(x)$ and $\alpha_2(x)$ are defined by exponential functions as the expressions (2) and (3). According to the material parameter for a super alloy (Rene-41)/Zirconia FGM layer in Ref. [2], we have $\ln(E_3/E_1)=0.37498$, $\ln(\lambda_3/\lambda_1)=2.5014$ and $\ln(\alpha_3/\alpha_1)=$ 0.51283 (e.g., E_3/E_1 =1.455, $\lambda_3/\lambda_1=12.2$ and $\alpha_3/\alpha_1=$ 1.67). When the structure undergoes a uniform temperature change, e.g., $T_1(x)=T_2(x)=T_3(x)=T_{01}=T_{02}=T$, the thermal stress distribution and the TSIFs are shown in Fig.2 and Fig.3. From Fig.2 and 3, it can be found that the resent results are very similar to those of Fig.3 and Fig.8 in Erdogan and Wu [2].

Next, we will consider a three-layered structure with $h_1=h_2=h_3=h/3$. Consider an embedded crack with it center located at the center of Layer 2, e.g., $c_0/h=0.5$. The material parameters of the functionally graded interfacial layer is same to the above (Rene-41)/Zirconia FGM layer. Assume the structure undergoes a steady-state heat conduction with the surface temperatures $T_{02}/T_0=1$ and $T_{01}\neq T_{02}$. The corresponding TSIFs are shown in Fig.4. It is found that the TSIFs increase with the increasing of T_{02}/T_0. Moreover, the TSIFs decrease with the increase of the crack length.

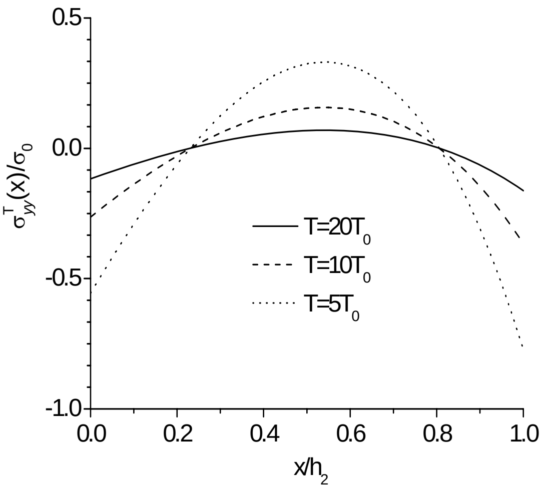

Fig. 2. Normalized thermal stress distribution in a (Rene-41)/Zirconia FGM layer

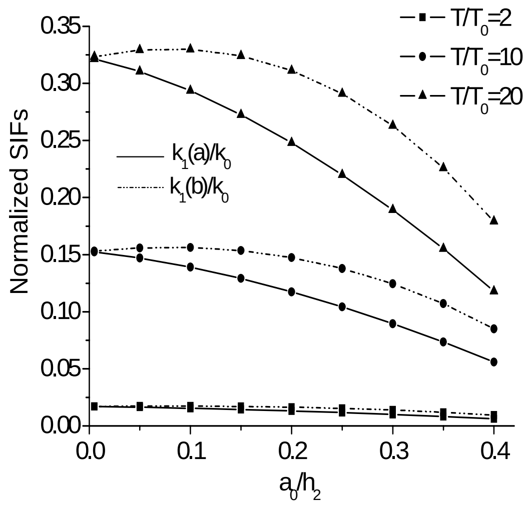

Fig. 3. Normalized TSIFs for an embedded crack in a (Rene-41)/Zirconia FGM layer

Figure 5 shows the influences of the crack length and modulus ratio on the normalized TSIFs for an embedded crack in a 3-layered structure. We assume $h_1=h_2=h_3=h/3$, $c_0/h=0.5$, $T_{02}/T_0=1$, $T_{01}/T_0=10$, $\lambda_3/\lambda_1=12.2$ and $\alpha_3/\alpha_1=$ 1.67. From Fig.5, it is found that when the modulus ratio E_3 / E_1 =1.455 and 2.0, the corresponding TSIFs exhibit great difference. Therefore, the nonhomogeneous parameters have great influence on the TSIFs.

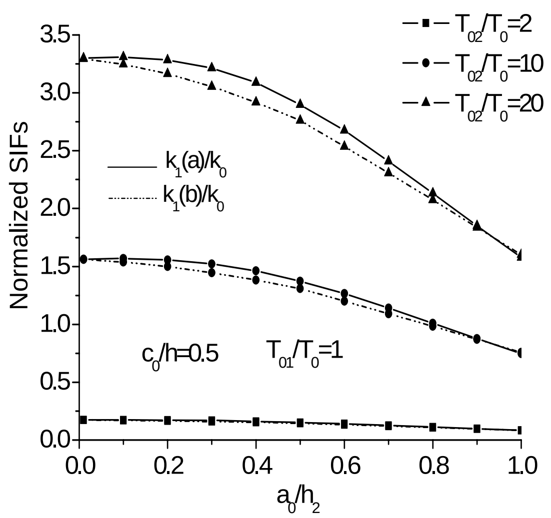

Fig. 4. Influences of the crack length and temperature variations on the normalized TSIFs for an embedded crack in a 3-layered structure

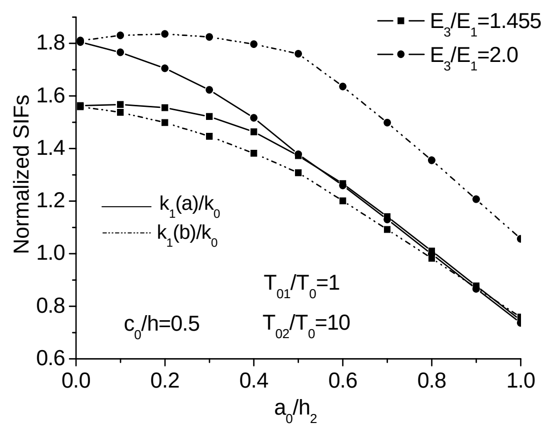

Fig. 5. Influences of the crack length and modulus ratio on the normalized TSIFs for an embedded crack in a 3-layered structure

4 Conclusions

In this paper, the thermal stress intensity factors for a crack in a functionally graded layered structure are analyzed. A crack intersecting the interface is considered. The influences of the nonhomogeneous thermal and mechanical properties and crack length on the thermal stress intensity factor are analyzed. Typically, the variations of TSIFs under different temperature changes and modulus ratios are depicted when the crack intersects the interface. It is found the influences of the temperature changes and modulus ratio on the thermal stress intensity factors are significant.

Acknowledgments

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