MODELS TO PREDICT PLY CRACKING AND ITS EFFECTS ON LAMINATE PROPERTIES

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Abstract

This paper summarises a methodology that can be used to predict ply crack formation in general symmetric laminates that are subject to general in-plane loading and thermal residual stresses. The basic methodology assumes that ply cracks from in a single orientation. A description is also given of how the methodology may be extended by the use of homogenization techniques to deal with ply cracking in any orientation. For cross-ply laminates references are given that describe a similar methodology that can be used to predict ply crack formation in an unsymmetrical laminate subject to biaxial combined axial stresses and bending moments per unit cross-sectional area, and thermal residual stresses.

1 Introduction

This paper will summarise the methodology that has been developed to predict damage formation in general symmetric laminates subject to general in-plane loading and thermal residual stresses. It has already been evaluated in the World-wide Exercise on Failure of Composites [1], and was the only methodology in that phase of the Exercise that considered sub-critical damage formation and the effect of some damage mechanisms on failure. Some extensions of the approach have been developed and these will also be described in this paper.

2 Specific objectives and assumptions

The first specific objective of this paper is to indicate the methods used for predicting the effective thermo-elastic constants of undamaged fibre reinforced composites, in both UD and laminated forms, from fibre and matrix properties, assuming that there is a good bond between fibre and matrix, and at the interfaces between the plies of laminates. For laminates, predictions can be made using ply data predicted from fibre and matrix properties based on these models, or from experimentally measured data.

The principal damage modes that need to be modelled reliably are: ply cracking, delamination and fibre failure, including fibre/matrix debonding associated with fibre fractures. This paper will concentrate on ply cracking, which is the first damage mode that is encountered during the monotonic and cyclic loading of a laminate. When laminated composites are loaded sufficiently, microstructural damage in the form of ply cracking occurs, which modifies the effective thermo-elastic constants of the laminate. The second objective of this paper is to indicate how changes in many of the effective thermo-elastic constants resulting from ply crack formation can be represented in terms of a single damage parameter. The third and major specific objective is to indicate how the thermo-elastic constants may be predicted as a function of the applied loading.

The complexity of the analysis to be discussed in this paper is such that its practical implementation must be through the use of a user-friendly software application. The codes have been integrated into a software tool known as PREDICT, which is one module of the NPL-developed software tool CoDA that is commercially available through Anaglyph:

www.npl.co.uk/npl/cmmt/cog/cmmthm098.html

A key feature of the software application is the speed at which problems can be defined, and at which computations can be carried out.

For application in structures, methods are needed to account for complex loading such as: in-plane biaxial and shear loading, through-thickness loading (of importance at bolt holes), and out-of-plane bending, including anticlastic effects. Although models for all these loading modes have been developed, only general in-plane strain will be considered here in detail. Methods are also needed to account for thermal residual stresses and those due to moisture ingress (usually non-uniform), although thermal residual stress effects only are considered in this paper.

The assumptions on which the methodology is based are:

1. for UD composites the fibre and matrix are made of different linear thermo-elastic materials (possibly anisotropic), which are perfectly bonded together everywhere on fibre/matrix interfaces. The cross-section of the fibres is assumed to be uniform and circular. The fibres are assumed to be uniformly distributed having a hexagonal packing structure leading to transverse isotropic ply properties.
2. laminates are symmetric (about their mid-planes) and constructed of any number of plies that can be made of different anisotropic materials, and are perfectly bonded together at all points on the inter-laminar interfaces.
3. when considering general symmetric laminates, the laminates can be unbalanced.
4. there exists a stress-free temperature T_o at which UD composites or laminates, when in an unloaded state, are free of any internal stresses. The temperature difference $\Delta T = T - T_o$, where T is the current temperature, appears in the stress/strain relations and, when non-zero, leads to thermal residual stresses in the composite, when it is not subject to mechanical loading.
5. any ply cracks in the plies of laminates are assumed to be planar and fully developed (i.e. the entire cross-section of a ply is cracked). The ply crack surfaces are assumed to be stress-free.

3 Undamaged composites

Given the bewildering number of ply materials that can be made from the wide variety of available fibres and matrices, and the large number of laminate lay-ups that can be constructed from UD plies, it is obvious that modelling undamaged composites from fibre and matrix properties is a key requirement that has already been achieved and used extensively in design. The objective here is to indicate convenient methods for calculating the effective thermo-elastic constants of undamaged UD composites, some of which are almost impossible to measure, for input into the modelling of laminate behaviour. Concentric cylinder models and variational techniques have been used extensively to predict the properties of UD composites (see review by Hashin [2]).

UD fibre reinforced composites are realistically modelled as transverse isotropic solids since fibre arrangements in real materials, while exhibiting non-uniform fibre spacings, are such that in the plane normal to the fibre direction the properties can be regarded as being more or less isotropic. Regular hexagonal arrays of aligned fibres in a composite lead exactly to transverse isotropic properties, while square and diamond arrays clearly do not have such properties. Transverse isotropic solids are characterised by seven independent thermo-elastic constants, which are denoted *in italics* by: E_A - axial Young's modulus, E_T - transverse Young's modulus, ν_A - axial Poisson's ratio, ν_T - transverse Poisson's ratio, μ_A - axial shear modulus, α_A - axial thermal expansion coefficient, α_T - transverse thermal expansion coefficient. Methods of estimating the values of these properties for undamaged UD composites will now be given.

For UD fibre reinforced composites, the concentric two-cylinder model has been used extensively. The effective axial Young's modulus, the axial Poisson's ratio and the transverse bulk modulus are given by

$$E_A = \mathrm{V_f E_A^f} + \mathrm{V_m E_A^m} + 2\lambda(\nu_A^m - \nu_A^f)^2 \mathrm{V_f V_m}\,, \quad (1)$$

$$\nu_A = \mathrm{V_f}\nu_A^f + \mathrm{V_m}\nu_A^m - \frac{\lambda}{2}(\nu_A^f - \nu_A^m)\left(\frac{1}{k_T^f} - \frac{1}{k_T^m}\right)\mathrm{V_f V_m}\,, \quad (2)$$

$$\frac{1}{k_T} = \frac{\mathrm{V_f}}{k_T^f} + \frac{\mathrm{V_m}}{k_T^m} - \frac{\lambda}{2}\left(\frac{1}{k_T^f} - \frac{1}{k_T^m}\right)^2 \mathrm{V_f V_m}\,, \quad (3)$$

$$\text{with} \quad \frac{1}{\lambda} = \frac{1}{2}\left[\frac{1}{\mu_T^m} + \frac{\mathrm{V_f}}{k_T^m} + \frac{\mathrm{V_m}}{k_T^f}\right]. \quad (4)$$

As the values of E_A, ν_A and k_T are known from (1-3), the value of $(1-\nu_T)/E_T$ can be then be calculated. As the UD composite is being modelled as a transverse isotropic solid the following relationship holds $E_T = 2\mu_T(1+\nu_T)$ but $E_A \neq 2\mu_A(1+\nu_A)$. Bounds for the transverse shear modulus can be estimated using variational techniques [2], and the following result corresponds exactly to the realistic bound where the matrix surrounds the fibre rather than the converse

$$\mu_T = \mathrm{V_f}\mu_T^f + \mathrm{V_m}\mu_T^m - \lambda_1\left(\mu_T^f - \mu_T^m\right)^2 \mathrm{V_f V_m}\,, \quad (5)$$

$$\text{with} \quad \frac{1}{\lambda_1} = V_m \mu_T^f + V_f \mu_T^m + \frac{k_T^m \mu_T^m}{k_T^m + 2\mu_T^m} \; . \qquad (6)$$

It is then possible to determine the values of E_T and ν_T. The axial shear modulus μ_A is given by the formula

$$\mu_A = V_f \mu_A^f + V_m \mu_A^m - \lambda_2 (\mu_A^f - \mu_A^m)^2 V_f V_m \, , \qquad (7)$$

$$\text{with} \quad \frac{1}{\lambda_2} = \mu_A^m (1 + V_f) + \mu_A^f V_m \; . \qquad (8)$$

On defining $\hat{\alpha}_T = \alpha_T + \nu_A \alpha_A$, the axial and transverse thermal expansion coefficients predicted by the concentric cylinder model are given by

$$\begin{aligned} E_A \alpha_A &= V_f E_A^f \alpha_A^f + V_m E_A^m \alpha_A^m \\ &\quad + 2\lambda (\nu_A^m - \nu_A^f) \left(\hat{\alpha}_T^m - \hat{\alpha}_T^f \right) V_f V_m \, , \end{aligned} \qquad (9)$$

$$\begin{aligned} \hat{\alpha}_T &= \hat{\alpha}_T^f V_f + \hat{\alpha}_T^m V_m \\ &\quad + \frac{\lambda}{2} \left(\frac{1}{k_T^f} - \frac{1}{k_T^m} \right) \left(\hat{\alpha}_T^m - \alpha_T^f \right) V_f V_m \, . \end{aligned} \qquad (10)$$

The UD properties are used to estimate the effective thermo-elastic properties of undamaged laminates using standard methods.

4 Modelling ply cracking

Before summarising the approaches to be made when developing physically based methods for predicting the formation of ply cracking in laminated composites, it is useful to address key issues that must be considered. In practice, composite structures are often in the form of laminates so that the effects of the weak transverse properties of UD composites can be avoided to some extent. During the loading of such laminates, the first damage mode that is often encountered is that of ply cracking where the matrix cracks in a ply in a direction that is parallel to the fibres. Such cracks form progressively during loading and lead to a reduction in laminate properties, and to a non-linear stress-strain response (i.e. strain softening). One rather crude approach is to perform a stress analysis on an undamaged structure using homogenised properties for the laminate and then to apply a quadratic stress-based failure criterion to the laminate. Another approach applies the failure criterion to individual plies in order to identify the ply that first cracks. The properties of this ply are then discounted (i.e. appropriate properties are reduced to zero) even though a single ply crack hardly affects the properties of the laminate, while the ply discount approach leads to a maximum loss of properties. A consequence of this stress-based approach is that when the methodology is applied to a laminate subject to traction boundary conditions, if all the dimensions of the laminate are doubled, the failure stress is unchanged, although the failure load would be doubled. This characteristic is not observed when damage forms in laminates. Ply crack suppression occurs when the thickness of the plies is reduced sufficiently. This phenomenon is the well-known ply thickness effect on ply crack formation in laminates.

A methodology developed for multiple ply cross-ply laminates subject to in-plane loading has been extended to the practically important case of general symmetric laminates. A theoretical framework has been developed [3] that shows how ply crack formation can be predicted in general symmetric laminates subject to general in-plane loading. The analysis shows that a stress transfer model is needed only to predict the dependence of the effective thermo-elastic constants on crack density. Reference [3] provides the details of the theoretical modelling that has been used to develop a reliable technique for predicting stress transfer between the neighbouring plies of a general symmetric laminate that may have uniformly spaced ply cracks in some or all of the plies having a 90^o orientation with respect to the principal loading direction (i.e. axial direction). The methodology assumes that there is perfect bonding between the plies of the laminate, and the analysis is developed so that account can be taken of the effects of uniform temperature changes. The technique is basically analytical, but because of the resulting complexity associated with solving systems of ordinary differential equations, the analysis must be handled numerically when making predictions of the behaviour of the laminate. In applications of the analysis, ply refinement techniques must again be used where each ply of the laminate is subdivided into layers having the same properties in order that through-thickness variations of the stress and displacement components can be taken into account. Such through-thickness variations are of importance in the neighbourhoods of ply cracks and accuracy is improved if they are modelled realistically.

Indeed, the key aspect of the stress-transfer models is their accuracy, which enables their reliable application in engineering design procedures for dealing with ply crack formation in complex laminates subject to complex loading. Shear-lag models do not possess this reliability and are *not* recommended for application in engineering design. The models have been implemented in software (see

Section 2) and can be used to predict the effective thermoelastic constants of laminates having ply cracks in a single orientation.

The rational predictive framework [3-5] for general symmetric laminates subject to general in-plane loading is based on energy principles where, for quasi-static conditions, the energy available for crack formation is balanced by the energy needed to form new ply crack surfaces. The framework is developed in terms of the effective thermo-elastic properties of the damaged laminate, and it has been shown that the criteria for first ply failure and progressive ply crack formation can be written down simply and solely in terms of parameters that are defined at the macroscopic laminate level even though the methodology has analysed behaviour at the individual ply level, and at the fibre/matrix level for conditions of perfect fibre/matrix bonding. The framework is derived without having to introduce a stress transfer model that accounts for localised stress redistribution in the laminate arising from ply crack formation. Such a model is needed only to calculate the effective thermo-elastic constants of a general symmetric laminate containing ply cracks, including accounting for non-uniform ply crack distributions [6].

The stress transfer model is based on stress and displacement fields, defined at all points within a general symmetric laminate subject to general in-plane loading, that satisfy exactly:

- equilibrium equations,
- compatibility equations,
- interfacial continuity conditions for both tractions and displacements,
- external boundary conditions involving tractions,
- all stress/strain relations except for: in-plane axial, transverse and shear equations that are satisfied in an average sense.

The in-plane stress/strain relations and displacement boundary conditions cannot be satisfied by the representation, but can be satisfied if they are averaged through the thickness of each ply, or of each ply element when using ply refinement techniques to increase the accuracy of predictions. The resulting averaged equations are such that if the representation were to be used in a variational calculation based on the Reissner energy functional [7] denoted by J, then the satisfaction of the averaged stress/strain relations is sufficient to ensure that $\delta J = 0$, indicating that the model is one of high quality (see [8], Appendix A for details).

5 Effective stress-strain relations

Consider any symmetric multiple-ply laminate whose distribution of damage in each ply is effectively uniform at the macroscopic level such that the effective stress-strain relations of the damaged laminate may be expressed [3, eqns. (76-79)] in the same form as that for an uncracked laminate, namely

$$\varepsilon_t = \frac{\sigma_t}{E_t} - \frac{\nu_a}{E_A}\sigma - \frac{\nu_t}{E_T}\sigma_T - \frac{\lambda_t}{E_A}\tau + \alpha_t \Delta T, \qquad (11)$$

$$\varepsilon = -\frac{\nu_a}{E_A}\sigma_t + \frac{\sigma}{E_A} - \frac{\nu_A}{E_A}\sigma_T - \frac{\lambda_A}{E_A}\tau + \alpha_A \Delta T, \qquad (12)$$

$$\varepsilon_T = -\frac{\nu_t}{E_T}\sigma_t - \frac{\nu_A}{E_A}\sigma + \frac{\sigma_T}{E_T} - \frac{\lambda_T}{E_A}\tau + \alpha_T \Delta T, \qquad (13)$$

$$\gamma = -\frac{\lambda_t}{E_A}\sigma_t - \frac{\lambda_A}{E_A}\sigma - \frac{\lambda_T}{E_A}\sigma_T + \frac{\tau}{\mu_A} + \alpha_S \Delta T, \qquad (14)$$

where ΔT is the temperature difference, E_A is the effective axial Young's modulus for a damaged laminate, and similarly for the other properties. In (11-14), the parameters σ_t, σ, σ_T and τ denote the in-plane effective applied through-thickness, axial, transverse and shear stresses. For specified values of σ_t, σ, σ_T, τ and ΔT, the quantities ε_t, ε, ε_T and γ are respectively the in-plane through-thickness, axial, transverse and shear strains of the damaged laminate. The parameters λ_t, λ_A and λ_T are ratios indicating the degree of shear coupling, and the parameter α_S is a thermal expansion coefficient governing the amount of shear strain that can arise when the temperature is changed. These parameters are zero for the special case of damaged cross-ply laminates, and for undamaged balanced general symmetric laminates.

It is understood that the values of the thermo-elastic constants in (11-14) may be determined by detailed stress analysis for any distribution of fully developed ply cracks. Indeed, calculations using the methods described in [3] indicate that for the case of uniform crack densities, stress-strain relations of the form (11-14) are obeyed. It has been shown in [6] that the stress-strain relations are valid also for non-uniform ply crack distributions. When the laminate is undamaged, the thermo-elastic constants are written with a superscript 'o': thus E_A^o denotes the axial Young's modulus for an undamaged laminate, and similarly for the other thermo-elastic constants.

It follows from (14) that

$$\tau = \mu_A \gamma + \frac{\mu_A}{E_A}\left[\lambda_t \sigma_t + \lambda_A \sigma + \lambda_T \sigma_T - E_A \alpha_S \Delta T \right]. \tag{15}$$

Substitution in (11-13) then leads to the following reduced stress-strain equations that are of the same form as those valid for in-plane biaxial loading modes [3, eqns. (85-87)]

$$\bar{\varepsilon}_t \equiv \varepsilon_t + \mu_A \frac{\lambda_t}{E_A}\gamma = \frac{\sigma_t}{\bar{E}_t} - \frac{\bar{\nu}_a}{\bar{E}_A}\sigma - \frac{\bar{\nu}_t}{\bar{E}_T}\sigma_T + \bar{\alpha}_t \Delta T, \tag{16}$$

$$\bar{\varepsilon} \equiv \varepsilon + \mu_A \frac{\lambda_A}{E_A}\gamma = -\frac{\bar{\nu}_a}{\bar{E}_A}\sigma_t + \frac{\sigma}{\bar{E}_A} - \frac{\bar{\nu}_A}{\bar{E}_A}\sigma_T + \bar{\alpha}_A \Delta T, \tag{17}$$

$$\bar{\varepsilon}_T \equiv \varepsilon_T + \mu_A \frac{\lambda_T}{E_A}\gamma = -\frac{\bar{\nu}_t}{\bar{E}_T}\sigma_t - \frac{\bar{\nu}_A}{\bar{E}_A}\sigma + \frac{\sigma_T}{\bar{E}_T} + \bar{\alpha}_T \Delta T, \tag{18}$$

where $\bar{\varepsilon}_t$, $\bar{\varepsilon}$ and $\bar{\varepsilon}_T$ can be interpreted as the axial and transverse strains for a damaged laminate that is constrained so that the shear strain is zero, and where [3, eqns. (88-96)]

$$\frac{1}{\bar{E}_t} = \frac{1}{E_t}\left(1 - \lambda_t^2 \frac{E_t \mu_A}{E_A^2}\right), \tag{19}$$

$$\frac{1}{\bar{E}_A} = \frac{1}{E_A}\left(1 - \lambda_A^2 \frac{\mu_A}{E_A}\right), \tag{20}$$

$$\frac{1}{\bar{E}_T} = \frac{1}{E_T}\left(1 - \lambda_T^2 \frac{E_T \mu_A}{E_A^2}\right), \tag{21}$$

$$\frac{\bar{\nu}_t}{\bar{E}_T} = \frac{1}{E_T}\left(\nu_t + \lambda_t \lambda_T \frac{E_T \mu_A}{E_A^2}\right), \tag{22}$$

$$\frac{\bar{\nu}_a}{\bar{E}_A} = \frac{1}{E_A}\left(\nu_a + \lambda_t \lambda_A \frac{\mu_A}{E_A}\right), \tag{23}$$

$$\frac{\bar{\nu}_A}{\bar{E}_A} = \frac{1}{E_A}\left(\nu_A + \lambda_A \lambda_T \frac{\mu_A}{E_A}\right), \tag{24}$$

$$\bar{\alpha}_t = \alpha_t + \lambda_t \frac{\mu_A}{E_A}\alpha_S, \tag{25}$$

$$\bar{\alpha}_A = \alpha_A + \lambda_A \frac{\mu_A}{E_A}\alpha_S, \tag{26}$$

$$\bar{\alpha}_T = \alpha_T + \lambda_T \frac{\mu_A}{E_A}\alpha_S. \tag{27}$$

The relations (16-18) are of exactly the same form as the stress-strain relations for symmetric cross-ply laminates deforming without bending.

For general symmetric laminates subject to deformations for which the shear strain is constrained to be zero, it is convenient to introduce the macroscopic damage parameter D defined by

$$D = \frac{1}{\bar{E}_A} - \frac{1}{\bar{E}_A^o}. \tag{28}$$

When the laminate is undamaged $D = 0$. It has been shown [3, 8] that the following relations can be satisfied for *any* state of ply cracking in the 90° plies

$$\frac{1}{\bar{E}_t} - \frac{1}{\bar{E}_t^o} = \left(\bar{k}'\right)^2 D, \tag{29}$$

$$\frac{1}{\bar{E}_A} - \frac{1}{\bar{E}_A^o} = D, \tag{30}$$

$$\frac{1}{\bar{E}_T} - \frac{1}{\bar{E}_T^o} = \bar{k}^2 D, \tag{31}$$

$$\frac{\bar{\nu}_t^o}{\bar{E}_T^o} - \frac{\bar{\nu}_t}{\bar{E}_T} = \bar{k}\,\bar{k}' D, \tag{32}$$

$$\frac{\bar{\nu}_A^o}{\bar{E}_A^o} - \frac{\bar{\nu}_A}{\bar{E}_A} = \bar{k} D, \tag{33}$$

$$\frac{\bar{\nu}_a^o}{\bar{E}_A^o} - \frac{\bar{\nu}_a}{\bar{E}_A} = \bar{k}' D, \tag{34}$$

$$\bar{\alpha}_t - \bar{\alpha}_t^o = \bar{k}' \bar{k}_1 D, \tag{35}$$

$$\bar{\alpha}_A - \bar{\alpha}_A^o = \bar{k}_1 D, \tag{36}$$

$$\bar{\alpha}_T - \bar{\alpha}_T^o = \bar{k}\,\bar{k}_1 D. \tag{37}$$

In the above inter-relationships the parameters $\bar{k}$, $\bar{k}'$ and $\bar{k}_1$ are constants for undamaged laminates defined in reference [3, 8].

6 Predicting damage formation

Consider a general symmetric laminate that already contains fully developed ply cracks in the 90° plies characterised by the damage state 1. Then consider the simultaneous formation of new fully developed ply cracks such that the resulting damage state is characterised by the damage state 2. The locations of the cracks are assumed to be such that the overall deformation of the laminate is governed by the effective stress/strain equations. This means that the ply crack distribution is regarded as being sufficiently uniformly distributed for shear and out-of-plane bending deformations to be negligible. The new ply crack surfaces are assumed to form quasi-statically under conditions of fixed effective triaxial applied tractions (without shear) and fixed temperature. From energy balance considerations and the fact that kinetic energy is never negative, the criterion for ply crack formation under these conditions has the form [3, 8]

$$\Delta\Gamma + \Delta G < 0\,, \tag{38}$$

where the energy absorbed in volume V of laminate by the formation of the new ply cracks is given by

$$\Delta\Gamma = V\left[\Gamma_2 - \Gamma_1\right]. \tag{39}$$

In (39), Γ_1 and Γ_2 denote, for damage states 1 and 2, the energy absorbed in unit volume during the formation of the ply crack surfaces in the 90° plies. The energy absorbed per unit volume is taken as the total area of crack surface formed in unit volume of laminate multiplied by the fracture energy for ply cracking. The corresponding change of Gibbs free energy in region V of the laminate is

$$\Delta G = \int_V \left[g_2 - g_1\right] dV\,, \tag{40}$$

where g_1 and g_2 are the non-uniform distributions of Gibbs free energy per unit volume for the two damage states. The Gibbs free energy is used in (38) as the applied stresses and temperature are regarded as independent variables. If external displacements were specified rather than applied stresses, then the Helmholtz free energy would replace the Gibbs free energy. The Gibbs free energy corresponds to the complementary energy that is used when carrying out energy balance calculations in the absence of thermal residual stresses.

It follows from (38-40) that ply crack formation associated with the change in damage from state 1 to state 2 is governed by the inequality

$$P_2 < P_1\,, \tag{41}$$

where P is the total energy defined by

$$P \equiv \Gamma + \frac{1}{V}\int_V \left[g - g_0\right] dV\,. \tag{42}$$

It has been shown [3, eqn. (153)] that

$$\begin{aligned}\langle g\rangle \equiv \frac{1}{V}\int_V g\,dV = \langle g_0\rangle - \frac{1}{2}D\left[\bar{s} - \bar{\sigma}_c\right]^2 \\ - \frac{1}{2}\mu_A\gamma^2 + \frac{1}{2}\mu_A^o\{\gamma^o\}^2\,,\end{aligned} \tag{43}$$

where $\langle g_0\rangle$ is the value of $\langle g\rangle$ when the laminate is undamaged, and where

$$\bar{s} = \bar{k}'\sigma_t + \sigma + \bar{k}\,\sigma_T\,, \tag{44}$$

is the effective applied stress and where $\bar{\sigma}_c = -\bar{k}_1\Delta T$ is the ply crack closure stress when a general symmetric laminate is subject to uniaxial loading in the axial direction normal to the planes of the ply cracks. The applied stress dependence appears as a perfect square involving a linear combination of the applied stresses. It can also be shown [3, eqn. (148-150)] that

$$\bar{\varepsilon}_t - \bar{\varepsilon}_t^o = \bar{k}'D\left[\bar{s} - \bar{\sigma}_c\right]\,, \tag{45}$$

$$\bar{\varepsilon} - \bar{\varepsilon}^o = D\left[\bar{s} - \bar{\sigma}_c\right]\,, \tag{46}$$

$$\bar{\varepsilon}_T - \bar{\varepsilon}_T^o = \bar{k}\,D\left[\bar{s} - \bar{\sigma}_c\right]\,. \tag{47}$$

The result (43) may then be written in the following equivalent form

$$\begin{aligned}\langle g\rangle - \langle g_0\rangle + \frac{1}{2}\mu_A\gamma^2 - \frac{1}{2}\mu_A^o\{\gamma^o\}^2 \\ = -\frac{1}{2}\left[\bar{\varepsilon} - \bar{\varepsilon}^o\right]^2 \Big/ \left(\frac{1}{\bar{E}_A} - \frac{1}{\bar{E}_A^o}\right).\end{aligned} \tag{48}$$

The use of (41), (42) and (48) leads to the first ply failure criterion [3, eqn. (155)]

$$\frac{\left[\bar{\varepsilon} - \bar{\varepsilon}^o\right]^2}{\dfrac{1}{\bar{E}_A} - \dfrac{1}{\bar{E}_A^o}} + \mu_A\gamma^2 - \mu_A^o\{\gamma^o\}^2 - 2\Gamma > 0\,, \tag{49}$$

where the energy absorption per unit volume Γ for length 2L of laminate appearing in (49) is given by [3, eqn.(156)]

$$\Gamma = \frac{h^{(90)}}{hL}\sum_{j=1}^{M}\delta_j\,\gamma_j^{(90)}\,, \tag{50}$$

where $2h^{(90)}$ is the total thickness of all 90° plies in the laminate having total thickness 2h, and where M is the number of potential cracking sites in 90° plies which are ordered in a regular way, e.g. from top to bottom in the plies which are taken in order from the centre of the laminate to the outside, symmetry about the mid-plane of the laminate being assumed. The quantity $2\gamma_j^{(90)}$ is the fracture energy for the j^{th} potential cracking site of the 90^o plies. The parameters δ_j describe the crack pattern in the laminate such that $\delta_j = 1$ if the j^{th} site of the 90^o ply is cracked and $\delta_j = 0$ otherwise.

It is emphasised that the approach described does not provide any information that indicates how the thermo-elastic constants depend upon the damage state. The detailed stress analysis described in [3, eqns.(49-79)] is needed to obtain this information.

It is easily shown [6] that the inter-relationships [3, eqns. (116), (123)] are also valid for non-uniform crack spacings and may be generalised for damage states 1 and 2 to the form [3, eqns. (160), (161)]

$$\frac{\bar{\nu}_t^{(2)}}{\bar{E}_T^{(2)}} - \frac{\bar{\nu}_t^{(1)}}{\bar{E}_T^{(1)}} = \bar{k}\left(\frac{\bar{\nu}_a^{(2)}}{\bar{E}_A^{(2)}} - \frac{\bar{\nu}_a^{(1)}}{\bar{E}_A^{(1)}}\right), \tag{51}$$

$$\frac{\bar{\nu}_A^{(2)}}{\bar{E}_A^{(2)}} - \frac{\bar{\nu}_A^{(1)}}{\bar{E}_A^{(1)}} = \bar{k}\left(\frac{1}{\bar{E}_A^{(1)}} - \frac{1}{\bar{E}_A^{(2)}}\right), \tag{52}$$

$$\frac{1}{\bar{E}_T^{(1)}} - \frac{1}{\bar{E}_T^{(2)}} = \bar{k}\left(\frac{\bar{\nu}_A^{(2)}}{\bar{E}_A^{(2)}} - \frac{\bar{\nu}_A^{(1)}}{\bar{E}_A^{(1)}}\right), \tag{53}$$

$$\bar{\alpha}_T^{(1)} - \bar{\alpha}_T^{(2)} = \bar{k}\left(\bar{\alpha}_A^{(1)} - \bar{\alpha}_A^{(2)}\right), \tag{54}$$

$$\frac{1}{\bar{E}_t^{(1)}} - \frac{1}{\bar{E}_t^{(2)}} = \bar{k}'\left(\frac{\bar{\nu}_a^{(2)}}{\bar{E}_A^{(2)}} - \frac{\bar{\nu}_a^{(1)}}{\bar{E}_A^{(1)}}\right), \tag{55}$$

$$\frac{\bar{\nu}_a^{(2)}}{\bar{E}_A^{(2)}} - \frac{\bar{\nu}_a^{(1)}}{\bar{E}_A^{(1)}} = \bar{k}'\left(\frac{1}{\bar{E}_A^{(1)}} - \frac{1}{\bar{E}_A^{(2)}}\right), \tag{56}$$

$$\frac{\bar{\nu}_t^{(2)}}{\bar{E}_T^{(2)}} - \frac{\bar{\nu}_t^{(1)}}{\bar{E}_T^{(1)}} = \bar{k}'\left(\frac{\bar{\nu}_A^{(2)}}{\bar{E}_A^{(2)}} - \frac{\bar{\nu}_A^{(1)}}{\bar{E}_A^{(1)}}\right), \tag{57}$$

$$\bar{\alpha}_t^{(1)} - \bar{\alpha}_t^{(2)} = \bar{k}'\left(\bar{\alpha}_A^{(1)} - \bar{\alpha}_A^{(2)}\right). \tag{58}$$

On using (51-58) it can be shown that the criterion for progressive cracking may be written in the compact form [3, eqn. (162)]

$$\left[\bar{\varepsilon}_2 - \bar{\varepsilon}_1\right]^2 \Big/ \left(\frac{1}{\bar{E}_A^2} - \frac{1}{\bar{E}_A^1}\right) + \mu_A^2\{\gamma_2\}^2 - \mu_A^1\{\gamma_1\}^2 - 2\left[\Gamma_2 - \Gamma_1\right] > 0, \tag{59}$$

which is consistent with the first ply failure condition (49). It must be emphasised that the result (59) takes full account of anisotropy and thermal residual stresses, and is valid for combined biaxial and shear loading conditions. The result is exact and simple in form. It is concluded that stress transfer models are needed only to estimate thermo-elastic constants of a cracked laminate. All other aspects of crack formation can be dealt with using the general framework described here. It is also emphasised that the criterion (59) is valid *only* if new cracks form for fixed values of σ_t, σ, σ_T, τ and ΔT.

The criteria (49) and (59), derived for first ply failure and progressive ply cracking respectively, would have rather complicated forms if expressed directly in terms of the parameters σ_t, σ, σ_T, τ and ΔT. The left hand sides of (49) and (59) would lead to quadratic forms in the variables σ_t, σ, σ_T, τ and ΔT that are difficult to solve in general. To simplify the problem, the effective applied stresses are now assumed to be proportional so that during loading

$$\sigma_t = \hat{A}\sigma, \quad \sigma_T = \hat{B}\sigma, \quad \tau = \hat{C}\sigma, \tag{60}$$

where $\hat{A}$, $\hat{B}$ and $\hat{C}$ are prescribed loading ratios. It is then a simple matter to derive a ply cracking criterion that is a quadratic equation for the axial stress σ.

When using (49) and (59) to investigate crack formation the fracture energies $2\gamma_j$ are taken at random from a statistical distribution of fracture energies (normal distribution assumed) and then allocated at random to the various potential fracture sites in the 90^o plies. The values of the applied stresses are then regarded as slowly increasing until the fracture condition (59) can be met at one of the potential fracture sites in the laminate. This means that ply crack formation, for every stage of loading, must be investigated at all remaining potential cracking sites in order to determine the location of the next crack to form. This iterative procedure, carried out using a computer, has to be repeated for every new crack that forms during loading. Such a procedure has been implemented in the PREDICT software system (see Section 2). It should be noted that the analysis described above for the prediction of ply cracking in general symmetric laminates has already been applied [9] in the International Failure Prediction Exercise [1].

7 Example predictions

It is useful to illustrate the capabilities of the model by including examples of the types of prediction that can be made. A 'two-dimensional' first ply failure envelope can be constructed that captures the effects of through-thickness loading, in-plane biaxial and shear loading on the formation of the first ply cracks. This is a new approach and needs to be applied to all orientations of the plies in one half of the laminate.

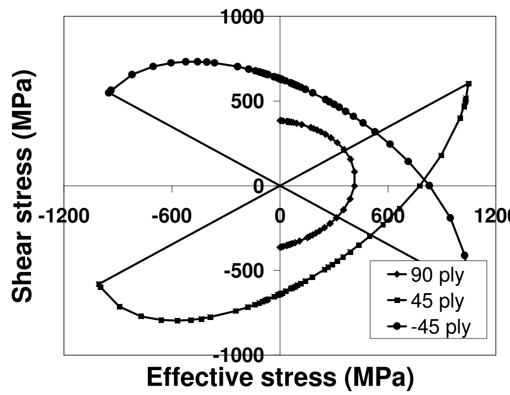

Fig. 1: First ply failure envelopes for a CFRP quasi-isotropic laminate for the case when $\Delta T = -90^oC$.

Fig.1 shows the first ply failure envelopes for the 90^o and the $\pm 45^o$ plies of a CFRP quasi-isotropic $[45/-45/0/90]_s$ laminate having thermal residual stresses ($\Delta T = -90^oC$). The envelope for the 0^o ply involves very large stresses and is not shown. The fracture energy for ply cracking is assumed to be 150 J/m^2. There are mixed-mode loading issues that also need to be considered. The appropriate fracture energy can be determined from a single test for a simple loading case, e.g. uniaxial tension.

The envelopes suggest that during loading ply cracking always occurs first of all in the 90^o ply. It is worth noting that the 90^o ply is located next to the mid-plane so that the total thickness of the 90^o ply is double that of any $\pm 45^o$ and 0^o plies. The envelopes are not closed, but terminate at points of ply crack closure. The lines drawn joining the closure points pass through the origin as shown in Fig.1.

Figure 2 shows the first ply cracking stress as a function of ply thickness for typical GRP and CFRP cross-ply laminates subject to uniaxial loading and a temperature difference ΔT = -85^{o}C. The total thickness of the laminate is always 4 mm so that as the ply thickness is reduced the number of plies progressively increases.

It is seen from Fig.2 that as the ply thickness reduces the first ply cracking stress increases (dramatically for the case of CFRP). Failure theories based on stress-based criteria would not predict this effect that has been observed in experiments.

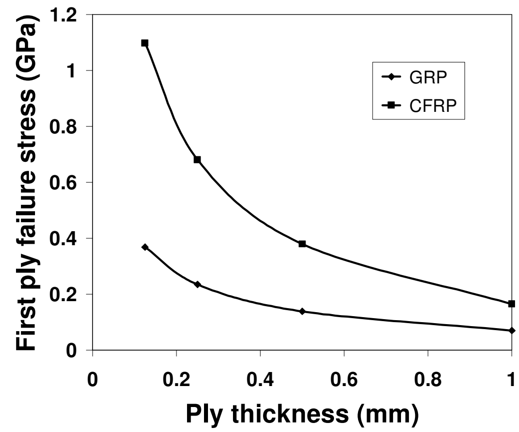

Figure 2: Effect of ply thickness on first ply cracking stress for CFRP and GRP laminates.

8 Ply cracking in any orientation

The model described in this paper has concerned ply cracking in a single orientation that corresponds to the 90^o plies of the laminate. As the model has been developed for general in-plane loading, it can be used to consider cracking in any other orientation simply by rotating the laminate so that the cracked ply becomes a 90^o ply, and by modifying the loading. If ply cracks form in more than one orientation, the determination of the stress and deformation distribution in the laminate is then a very difficult task. A pragmatic way forward is to develop an homogenisation method for general symmetric laminates, as described in detail in references [10, 11], that enables progressive ply crack formation to be predicted in any number of plies having a variety of orientations. The approach involves: i) the analysis of non-uniformly spaced discrete ply cracks having a single orientation, ii) a novel technique to homogenise the properties of the cracked ply so that discrete ply cracking can be analysed in plies having a different orientation, and iii) the use of energy based methods to predict the progressive formation of ply cracks in any number of plies during loading. The analysis takes full account of the effects of thermally induced residual stresses. A key feature of the approach is the inclusion of a shear coupling term in the stress-strain relations for homogenised plies that ensures that the homogenised laminate has exactly the same effective properties as the laminate having a ply with discrete cracks in place of one of the homogenised plies.

For laminates having only high-angle plies (e.g. orientations to loading axis that are greater than 60^o), it has been shown [10, 11] that good estimates of strength are obtained that show the correct dependence on ply thicknesses and ply lay-up. For laminates having low-angle plies (especially 0^o plies) an additional damage mode must be considered, i.e. fibre fracture. A method of dealing with this phenomenon has been developed and is described in reference [11].

9 Out-of-plane bending

In practical applications laminated composite materials are seldom subjected to just in-plane loading. Out-of-plane deformation is frequently encountered in conjunction with in-plane loading, and it is required to be able to predict damage formation in such situations. In addition to the complexity introduced by the inclusion of orthogonal bending modes of loading, there is the complication of dealing with ply cracking in multiple orientations of a general laminate (which need not in this case be symmetric). In order to adopt a tractable approach to this complex situation, an energy-based methodology has been developed [12, 13] that considers ply cracking in cross-ply laminates, which do not need to be symmetric, subject to combined in-plane biaxial loading and out-of-plane biaxial bending.

To date, a model has been developed that assumes that the fracture energy for ply cracking is not statistically distributed. The stress-state for first ply failure then has a precise value, and a corresponding characteristic uniform ply crack density is predicted by the model. Progressive damage involves the progressive doubling of the crack density at stress states that are determined by the model.

It is feasible for this model to be extended so that general symmetric laminates can be considered, and ply cracking in multiple orientations. This would involve adopting homogenisation techniques, which might impose limitations on the validity of the model. This is a topic for future investigation.

10 Some unresolved issues

1. The inclusion of delaminations in damage models is exceedingly difficult, and they are thought to have an important influence on progressive damage formation and ultimate failure. Realistic delamination modelling is likely to require numerical methods such as finite element and boundary element methods, due to the complexity of delamination shapes, problems of dealing with crack contacts that involve unknown contact areas and frictional effects.
2. The inclusion of fibre fracture as a damage mechanism is an important requirement. While it is feasible, when dealing with simple loading of fibres along and transverse to the fibre directions, the modelling of fibre failure in complex loading situations involving shear and bending, is an important requirement as it is very likely to have an effect on the failure of the laminate.
3. The methodology described in this paper is based on the application of an energy-based approach to damage formation. The procedure seems to be a unique approach, and it involves trial testing of all possible sites for damage formation or damage growth, and the selection of the site for damage formation that is the most energetically favourable. The approach has the advantage that both ply thickness and ply lay-up have an affect on damage formation, as observed experimentally. Other approaches based on a consideration of local stress states have been considered, and there is a need to determine the appropriate method for predicting damage formation and growth.

11 Conclusions

1. Because of their robustness and accuracy, the micro-mechanical models described in this paper, predicting undamaged ply properties from those of the fibre and matrix, have very good potential for being used seriously in the design of both composite materials, and of components made of these materials.
2. The energy-based framework described for predicting first ply failure and progressive ply crack formation in laminated composites is an excellent basis for the development of new design methods to avoid, or take proper account of, the micro-mechanisms of damage formation. The approach is exact within the assumptions made, and the methodology rigorously bridges the gap between ply related phenomena (e.g. ply cracking) and the macroscopic laminate behaviour of interest in engineering design.
3. The physically based damage models for laminates have a structure that is simple in form and is accurate. Very useful analytical results have been derived without a detailed knowledge

of the stress and deformation fields in damaged laminates.

4. A detailed stress transfer model is needed only to predict values for the thermo-elastic constants as a function of geometry and crack distribution. All other aspects of ply crack formation can be predicted from the general framework.
5. Methods can be used to predict the multi-axial non-linear behaviour of laminates when ply cracking occurs. Such methods have potential for integration into FEA codes for application to the prediction of first ply failure and progressive formation of ply cracks in the non-uniform stress fields that arise in composite components.
6. Physically based damage evolution methods can be based on the energy principles used in fracture mechanics. When statistical variability is included during the prediction of progressive damage formation, as required in practice, energy principles demand that all possible damage possibilities must be considered in order to determine the occurrence and order of energetically favourable damage events.

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