AN EXTENDED MORI-TANAKA'S MICROMECHANICS MODEL

Chun-Ron Chiang
Department of Power Mechanical Engineering
National Tsing Hua University, Hsin Chu, Taiwan

Keywords: *micromechanics, Mori-Tanaka's model, effective properties, composite, inclusion, Eshelby's tension*

Abstract

Mori-Tanaka's micromechanics model is re-derived from a new integral equation formulation in which a deviator field is introduced. It is shown that when the deviator is null, the original Mori-Tanaka's model is recovered. Due to the nature of unilateral variation of the deviator, the prediction of the effective moduli based on this formalism would never violate Hashin-Shtrikman's bounds. Composite systems of an incompressible matrix containing spherical rigid inclusions or containing spherical cavities are used to illustrate the present approach. The predictions are compared with those from other micromechanics models and the unique features of the present approach are discussed.

1 Introduction

A variety of approaches can be followed to estimate the effective elastic properties of composite materials [1,2]. Among them, Mori-Tanaka's model has received wide attention for its simplicity and easiness in applications [3-5]. Particularly, when the coupling-properties (elastic-electric, elastic-electric-magnetic, etc.) are involved [6], such an explicit model for the prediction of physical properties based on the micro-geometry and micromechanics considerations provides a very cost-effective and time-saving design tool for the development of new composites. However, some limitations of this model have been noted [7].

A theory for the estimation of the effective properties of composites consisting of dispersed inclusions in a matrix is developed in this article. It will be shown that when the inclusions are indistinguishable and point-like particles, the theoretical predictions are identical to Mori-Tanaka's results. Since the derivation of the present theory is quite different from the traditional route, the theory provides a new perspective to look at this type of micromechanics model. In fact, by certain generalizations, the limitations of the original Mori-Tanaka's model can be completely removed. The prediction of the shear modulus of an incompressible isotropic matrix containing spherical cavities and rigid inclusions of uniform size would be used to illustrate this point.

2 Integral Equation Formulation

Integral equation formulation for the determination of elastic fields in a heterogeneous medium has been used by many authors [8,9]. The strain distribution in the composite can be expressed by

$$\varepsilon_{ij}(x) = \varepsilon^0_{ij} - \int \Gamma_{ijkl}(x,x')\left(L_{klmn}(x') - L^0_{klmn}\right)\varepsilon_{mn}(x')dx' \quad (1)$$

where Γ is the second derivative of Green's function; ε^0, L, and L^0 are the applied strain, elastic stiffness of the constituent phase (at that location) of the composite and elastic stiffness of a reference medium, respectively. When the composite material under consideration is composed of a continuous phase and isolated inclusions of other phase, it is convenient to choose the matrix as the reference medium. Then, if x is in one of the inclusions (say Ω_1), (1) can be written as (assuming that there are N inclusions in a representative volume element V)

$$\varepsilon_{ij}(x) = \varepsilon^0_{ij} - \int_{\Omega_1} \Gamma_{ijkl}(x,x')\left(L_{klmn}(x') - L^{\mathrm{m}}_{klmn}\right)\varepsilon_{mn}(x')dx'$$

$$-\sum_{p=2}^{N} \int_{\Omega_p} \Gamma_{ijkl}(x,x')\left(L_{klmn}(x') - L^{\mathrm{m}}_{klmn}\right)\varepsilon_{mn}(x')dx' \quad (2)$$

On the other hand, if x is in the matrix, we have

$$\varepsilon_{ij}(x)=\varepsilon_{ij}^{0}-\sum_{p=1}^{N}\int_{\Omega_p}\Gamma_{ijkl}(x,x')\left(L_{klmn}(x')-L_{klmn}^{\mathrm{m}}\right)\varepsilon_{mn}(x')dx'$$

For the purpose of determination of the effective properties, only the mean strain and stress, not the detailed distribution, are needed. Taking the ensemble average of (2), the mean value of the strain in the inclusions can be written as

$$<\varepsilon\ >_{\mathrm{in}}=\varepsilon^{0}-<\frac{1}{\Omega_1}\int_{\Omega_1}dx\int_{\Omega_1}\Gamma(x,x')\left(L(x')-L^{\mathrm{m}}\right)\varepsilon(x')dx'>$$
$$-<\sum_{p=2}^{N}\frac{1}{\Omega_1}\int_{\Omega_1}dx\int_{\Omega_p}\Gamma\ (x,x')\left(L\ (x')-L^{\mathrm{m}}\right)\varepsilon(x')dx'> \qquad (3)$$

Here all the subscripts have been omitted for brevity. Similarly the equation governing the mean value of the strain in the matrix can be written as

$$<\varepsilon\ >_{\mathrm{m}}=\varepsilon^{0}$$
$$-<\sum_{p=1}^{N}\frac{1}{V-\Omega}\int_{V-\Omega}dx\int_{\Omega_p}\Gamma(x,x')\left(L(x')-L^{\mathrm{m}}\right)\varepsilon(x')dx'> \qquad (4)$$

where Ω denotes the union of all the inclusions. We will write (3) and (4) in a more compact forms as

$$<\varepsilon>_{\mathrm{in}}=\varepsilon^{0}-<I_1>-<I> \qquad (3')$$

$$<\varepsilon>_{\mathrm{m}}=\varepsilon^{0}-<M> \qquad (4')$$

Even with such a reduction, finding the average strains in both phases from Eq (3) and (4) is still a formidable task for general cases. In order to provide the background for the development of present theory, in the next section we shall briefly review two specific approaches.

3 Some Previous Models

In the following discussion, both the matrix and inclusions are assumed to be isotropic. The shape of the inclusion is assumed to be spherical and the dispersion of the inclusions in the matrix is in such a way that overall properties of the composite can be regarded as isotropic.

3.1 Dilute condition

When the concentration of the inclusions is low, the interaction among inclusions can be neglected. The last term in the right hand side of Eq. (3) can be dropped. It is well known that the strain inside an isolated inclusion is uniform [10], Eq.(3) turns out to be

$$<\varepsilon>_{\mathrm{in}}=\varepsilon^{0}-P\left(L^{\mathrm{in}}-L^{\mathrm{m}}\right)<\varepsilon>_{\mathrm{in}} \qquad (5)$$

where

$$P_{ijkl}(x)=\int_{\Omega_1}\Gamma_{ijkl}(x,x')dx' \qquad (6)$$

Since x is inside the inclusion, P is a constant tensor of rank four and depends only on the elastic moduli of the matrix. From Eq (5), the average strain of the inclusion can found as

$$<\varepsilon>_{\mathrm{in}}=\left(I+P(L^{\mathrm{in}}-L^{\mathrm{m}})\right)^{-1}\varepsilon^{0} \qquad (7)$$

while the average strain of the matrix can be readily determined by

$$<\varepsilon>_{\mathrm{m}}=(1-\mathrm{f}_{\mathrm{in}})^{-1}\left(\varepsilon^{0}-\mathrm{f}_{\mathrm{in}}<\varepsilon>_{\mathrm{in}}\right) \qquad (8)$$

where f_{in} denotes the volume fraction of the inclusion. Accordingly, the effective moduli can be determined as

$$L^{\mathrm{Dil}}=\left[I-\mathrm{f}_{\mathrm{in}}\left(I+P(L^{\mathrm{in}}-L^{\mathrm{m}})\right)^{-1}\right]L^{\mathrm{m}}$$
$$+\mathrm{f}_{\mathrm{in}}\left(I+P(L^{\mathrm{in}}-L^{\mathrm{m}})\right)^{-1}L^{\mathrm{in}} \qquad (9)$$

where superscript "Dil" has been used to signify Eq.(9) is under dilute condition. For isotropic phases and spherical inclusions we may write Eq. (9) more explicitly as

$$\frac{\kappa^{\mathrm{Dil}}}{\kappa^{\mathrm{m}}}=\frac{\kappa^{\mathrm{m}}+(\mathrm{f}_{\mathrm{in}}+\alpha)(\kappa^{\mathrm{in}}-\kappa^{\mathrm{m}})}{\kappa^{\mathrm{m}}+\alpha(\kappa^{\mathrm{in}}-\kappa^{\mathrm{m}})} \qquad (10)$$

$$\frac{\mu^{\mathrm{Dil}}}{\mu^{\mathrm{m}}}=\frac{\mu^{\mathrm{m}}+(\mathrm{f}_{\mathrm{in}}+\beta)(\mu^{\mathrm{in}}-\mu^{\mathrm{m}})}{\mu^{\mathrm{m}}+\beta(\mu^{\mathrm{in}}-\mu^{\mathrm{m}})} \qquad (11)$$

where κ and μ are bulk moduli and shear moduli respectively; while $\alpha=3\kappa^{\mathrm{m}}/(3\kappa^{\mathrm{m}}+4\mu^{\mathrm{m}})$ and $\beta=(6\kappa^{\mathrm{m}}+12\mu^{\mathrm{m}})/(15\kappa^{\mathrm{m}}+20\mu^{\mathrm{m}})$.

3.2 Mori-Tanaka's approach

When the volume fraction of the inclusions is not small, the proper account must be taken about the interaction among inclusions. Mori-Tanaka's mean field approach [3] can be followed to treat this interaction approximately [11,12]. There are several ways to look at this model [5]; the following derivation provides an alternative view.

The size of the composite is expected to be much larger than the size of the inclusion and in a representative volume element the number of the inclusions N is a large number, so we may take

$N \approx N-1$.When the inclusions can be treated as point-like particles which may occupy anywhere in the composite just like the material point of the matrix, then it follows that

$$< M > = < I > \tag{12}$$

By means of the Eq(12), canceling the term $< I >$ and $< M >$ in Eqs(3) and (4), it is found that

$$<\varepsilon>_{\rm in} = <\varepsilon>_{\rm m} - P\left(L^{\rm in} - L^{\rm m}\right) <\varepsilon>_{\rm in} \tag{13}$$

The above equation has exactly same form as Eq. (5) except that the mean strain in the matrix plays the role of the applied strain. Go through the similar procedure as discussed under the dilute condition, it is concluded that the effective elastic moduli can be predicted as

$$\frac{\kappa^{\rm MT}}{\kappa^{\rm m}} = \frac{\kappa^{\rm m} + [{\rm f_{in}} + \alpha(1-{\rm f_{in}})](\kappa^{\rm in} - \kappa^{\rm m})}{\kappa^{\rm m} + \alpha(1-{\rm f_{in}})(\kappa^{\rm in} - \kappa^{\rm m})} \tag{14}$$

$$\frac{\mu^{\rm MT}}{\mu^{\rm m}} = \frac{\mu^{\rm m} + [{\rm f_{in}} + \beta(1-{\rm f_{in}})](\mu^{\rm in} - \mu^{\rm m})}{\mu^{\rm m} + \beta(1-{\rm f_{in}})(\mu^{\rm in} - \mu^{\rm m})} \tag{15}$$

Note the differences between this pair of equations and the pair of Eqs.(10) and (11) are simply a factor of $(1-{\rm f_{in}})$ in front of α and β.

It is worth mentioning that the elastic moduli predicted by Mori-Tanaka's model correspond to the Hashin-Shtrikman's lower (upper) bound [12] if the inclusion is harder (softer) than the matrix [13].

4 The New Approach

4.1 Preliminary Considerations

Strictly speaking, Eq. (12) is not valid if the finite size of the inclusions is taken into account. However; due to linearity of the problem, Eq.(12) may be re-written in a more quantitative way

$$< M > - < I > = \Delta_{ij} = \Psi_{ijkl}\varepsilon^0_{kl} \tag{16}$$

where Δ will be referred as the deviator while Ψ called the deviator tensor which is a positive definite tensor of rank four and its exact determination requires complete information of how the inclusions are dispersed in the matrix. The resulting predictions of the moduli now would depend on this new introduced parameter. Obviously, the calculation of Ψ requires actual information about the distribution of the inclusions. The information is usually in terms of certain statistical parameters. But in some cases, a rough estimate may be sufficient as we discuss in the following example

We would like to consider an "old" problem: to determine the shear modulus of an incompressible isotropic matrix containing rigid spherical particle of uniform size. The prediction of the effective moduli can be written as

$$L^{\rm eff} = L^{\rm m}(I + (\frac{{\rm f_r}}{1-{\rm f_r}}I + {\rm f_r}\Psi)S^{-1}) \tag{17}$$

where ${\rm f_r}$ denotes the volume fraction of the rigid particles; *S* is the Eshelby's tensor. For this problem, only the shear modulus is involved, Eq.(17) can be regarded as a scalar equation. Writing Eq.(17) only up to second order terms for ${\rm f_r}$, we have

$$\mu^{\rm eff} = \mu^{\rm m}(1 + ({\rm f_r} + {\rm f_r^2} + {\rm f_r}\psi)\frac{5}{2}) \tag{18}$$

where *S* being equal to 2/5 has been used. Since ψ represents the average contribution of the strain enhancement local to the neighboring of the inclusions, it may be estimated as $\psi \approx {\rm f_r}$
Accordingly, Eq.(15) becomes

$$\mu^{\rm eff} = \mu^{\rm m}(1 + \frac{5}{2}{\rm f_r} + 5{\rm f_r^2}) \tag{19}$$

The prediction is almost identical to the result of Chen and Acrivos [14]

4.2 More Detailed Analysis

Central to the present problem is how to evaluate the deviator, i.e.

$$\Delta = \int \Gamma(0,x')[p^{\rm m}(x') - p^{\rm in}(x')][L^{\rm in} - L^{\rm m}]\varepsilon(x')dx' \tag{20}$$

which represents Eq.(16) in a more precise way. Here, $p^{\rm m}$ and $p^{\rm in}$ denote the probability density of finding an inclusion centered at *x'* on the condition that the origin is in the matrix phase and in the inclusion respectively. Statistical homogeneity has been assumed in such a representation. If isotropy of the composite is assumed further, then *x'* can be replaced by its radial distance and triple integral becomes simple one-dimension integral. However, exact evaluation of this term still needs much numerical effort which is avoided for the time being in the present investigation. In general, guided by this expression, it is proposed to write approximately for the deviator Δ as

$$\Delta = \phi\hat{P}(L^{\rm in} - L^{\rm m}) < \varepsilon >_{\rm in} \tag{21}$$

where ϕ is a statistical number (scalar quantity) describing the overall dispersion of the inclusions in the matrix; $\hat{P}$ being an isotropic tensor of rank four plays a similar role as P in Eq. (5). However, major difference between these two constant tensors should be noted: $\hat{P}$ is used to denote the weighted average (not in a strict sense) response of the exterior point of the inclusion while P refers to the interior point. Since a constant dilatation eigen strain of a spherical inclusion produce only dilatation-free deformation outside the inclusion, it is a reasonable approximation to assume that $\hat{P}_{iijj} = 0$. Accordingly, by using Eq.(21) as a correction term to the Mori-Tanaka model, the prediction of the bulk modulus is intact; only the prediction of the effective shear modulus needs modification. It is straightforward to find that

$$\frac{\mu^{EX}}{\mu^{m}} = \frac{\mu^{m} + [f_{in} + (\beta - \phi\hat{\beta})(1-f_{in})](\mu^{in} - \mu^{m})}{\mu^{m} + (\beta - \phi\hat{\beta})(1-f_{in})(\mu^{in} - \mu^{m})} \qquad (22)$$

To make further progress, it is still needed a way to evaluate $\phi\hat{\beta}$. The approach is motivated by the fact that the order of magnitude of β and $\hat{\beta}$ is the same. In fact, under certain circumstance they may have the same value. For example, when the matrix point can be regarded as an interior point inside a "homogeneous" inclusion which undergoes a deviatoric eigen strain ε*, the actual strain of the matrix point would be βε*. On the other hand, for an inclusion inside another inclusion of similar shape, when the outer inclusion (excluding the inner inclusion) undergoes an eigen strain ε*, the strain of the inner inclusion is zero [15]. In other word, the difference between these two quantities is simply given by βε*. Accordingly, in the following numerical computation, it is assumed that

$$\hat{\beta} = \beta \qquad (23)$$

Since the deviator tensor must be positive definite and the prediction be compatible with the dilute condition, the statistical number ϕ is taken as

$$\phi = \begin{cases} f_{in} & \text{if } \mu^{in} > \mu^{m} \\ -f_{in} & \text{if } \mu^{in} < \mu^{m} \end{cases} \qquad (24)$$

Apparently, Eq.(24) is valid only for small concentration. A more refinement may take the following form

$$\phi = \begin{cases} f_{in} - bf_{in}^{2} + \cdots & \text{if } \mu^{in} > \mu^{m} \\ -f_{in} + bf_{in}^{2} - \cdots & \text{if } \mu^{in} < \mu^{m} \end{cases} \qquad (25)$$

The coefficients associated with such a series expansion depend on the underlying randomness of microstructures of the composite. The determination of these coefficients requires extensive computational effort which is to be avoided. Besides, the approximate nature of Eq.(21) indicates such a refinement is unnecessary for the same order of accuracy. Therefore, only Eq.(24) will be used in the following numerical calculations (but see section 6 for some remarks).

5 Comparisons

A variety of approaches can be used to predict the effective moduli of a composite [1,2,7]. The following comparison is made only for three models: Self-Consistent (SC) Scheme [16], Mori-Tanaka's (MT) Model and the present Extended (EX) Mori-Tanaka's Model. To make the contrast more easily be visualized, only two extreme cases are under consideration. One case is incompressible matrix containing spherical rigid inclusions; the other case is associated with spherical cavities. Only the predictions of effective shear moduli are presented. Before presenting the results, it is worthwhile comparing the results with other "exact" predictions of reliable accuracy.

By expanding Eq.(22) in terms of f_{in} up to second order, using the fact that for incompressible matrix, β = 2/5, it is found that for rigid inclusions

$$\mu^{EX} = \mu^{m}(1 + \frac{5}{2}f_{r} + 5f_{r}^{2}) \qquad (26)$$

which is in agreement with Eq.(19). For original Mori-Tanaka's model, it is predicted that up to the second order

$$\mu^{MT} = \mu^{m}(1 + \frac{5}{2}f_{r} + \frac{5}{2}f_{r}^{2}) \qquad (27)$$

Comparing Eq.(26) and (27), it is concluded the prediction of present model is more faithful by using the results of Chen and Acrivos as standard values.

On the other hand, for cavities it is found

$$\mu^{EX} = \mu^{m}(1 - \frac{5}{3}f_{c}) \qquad (28)$$

The coefficient associated with the second order term is zero. It is interesting to observe that for original Mori-Tanaka's prediction is

$$\mu^{MT} = \mu^{m}(1-\frac{5}{3}f_c + \frac{10}{9}f_c^2) \tag{29}$$

While the extensive numerical calculation of Chen and Acrivos showed that

$$\mu^{CA} = \mu^{m}(1-\frac{5}{3}f_c + 0.5f_c^2) \tag{30}$$

Apparently, based on Eq.(30), for porous materials, the prediction of the present model tends to underestimate while that of Mori-Tanaka tends to overestimate.

In Table 1, results of composites containing rigid inclusions are displayed. It should be noted that the prediction of SC scheme becomes unbounded at f_r = 0.4. Results of composites containing spherical cavities are shown in Table 2. SC scheme predicts that effective shear modulus becomes zero at f_c = 0.5. Since the random-packing limit for uniform size of spheres is about 0.638, the calculation has accordingly been carried out up to the volume fraction of 0.64.

Table 1. Predictions of effective shear modulus of a composite containing spherical rigid inclusions based on different models

Volume fractions	μ^{SC}/μ^m	μ^{EX}/μ^m	μ^{MT}/μ^m
0.1	1.33	1.31	1.28
0.2	2.00	1.78	1.63
0.3	4.00	2.53	2.07
0.4	∞	3.78	2.67
0.5	∞	6.00	3.50
0.6	∞	10.38	4.75
0.64	∞	13.35	5.44

Table 2. Predictions of effective shear modulus of a composite containing spherical cavities based on different models

Volume fractions	μ^{SC}/μ^m	μ^{EX}/μ^m	μ^{EX*}/μ^m	μ^{MT}/μ^m
0.1	0.83	0.83	0.84	0.84
0.2	0.64	0.68	0.69	0.71
0.3	0.44	0.53	0.55	0.58
0.4	0.23	0.40	0.43	0.47
0.5	0	0.29	0.32	0.38
0.6	0	0.19	0.23	0.29
0.64	0	0.16	0.19	0.25

6 Remarks

Comparing the present predictions with those of Chen and Acrivos, it is found that using the approximate expression for the deviator (i.e. Eq.(21)), the prediction of rigid inclusion reinforced composites (i.e. Eq.(26)) is much better than that of porous materials (i.e. Eq.(28)). This is not too surprising because the strains of rigid inclusions are truly constant (all uniformly equal to zero), whereas the strains of spherical cavities are not so. Accordingly, Eq.(21) becomes less accurate for cavities. It is possible to improve the accuracy by introducing more precise strain state of the inclusion into the integral form of Eq.(21), either by actual numerical simulations or other analytical approaches. However, it can be improved by modifying Eq.(24) as another empirical expression such as

$$\phi = \frac{\mu^{in} - \mu^{m}}{\mu^{in} + \mu^{*}} f_{in} \tag{31}$$

where

$$\mu^{*} = \frac{3}{2}\left(\frac{1}{\mu^{m}} + \frac{10}{9\kappa^{m} + 8\mu^{m}}\right)^{-1} \tag{32}$$

By using Eq.(31), the prediction for rigid inclusions in an incompressible matrix is unchanged; while the prediction for cavities, up to second order term becomes

$$\mu^{EX*} = \mu^{m}(1-\frac{5}{3}f_c + \frac{10}{27}f_c^2) \tag{33}$$

It appears that the prediction is still lower than Eq.(30), but discrepancy has been greatly reduced. Substituting Eq.(31) into Eq.(22), the prediction for the whole spectrum of the volume fraction has been shown in Table 2.

Eq.(31) can also be used as a basis for the prediction of effective bulk modulus, explicitly it may be shown that

$$\frac{\kappa^{EX}}{\kappa^{m}} = \frac{\kappa^{m} + [f_{in} + (\alpha - \phi\alpha)(1-f_{in})](\kappa^{in} - \kappa^{m})}{\kappa^{m} + (\alpha - \phi\alpha)(1-f_{in})(\kappa^{in} - \kappa^{m})} \tag{34}$$

It is noted that when the shear moduli of the inclusion and matrix are the same, the prediction of Eq.(34) is the same as Mori-Tanaka's prediction which turns out to be exact solution of Hill [17].

It might be instructive to emphasize the main difference between the present integral formulation and the traditional formulation. By introduction of the deviator field Δ, the present formulation relies on the evaluation of a term like Eq.(20) which is absolutely convergent. On the other hand in the traditional formulation, non-absolutely converging integrals which arise because of long-range interactions of inclusions were encountered and such problems must be circumvented by the renormalization procedure or its equivalence [18].

Theoretically, a sample of infinite size is required for a proper definition of effective moduli. In practice, a sample of finite size cannot be avoided; the deviator then would be useful in this aspect as it provides a way for estimation of the influence of inclusion sizes. Without any calculation, it is ready to see that at the same concentration of inclusions the effective moduli of a composite system with small inclusion size tend to approach the original Mori-Tanaka's predictions; whereas those with larger inclusion size tend to deviate from the Mori-Tanaka's predictions.

7 Conclusions

By introducing a deviator field (a strain field deviates from Mori-Tanaka's model), a new integral equation formulation for the determination of effective properties of a composite material is developed. It has been shown that Mori-Tanaka's model is valid only if the deviator is null. This implies that original Mori-Tanaka's model can be applied to the case where the inclusions can be treated as point-like particles or under exceptional conditions that the deviator accidentally becomes zero. The deviator field depends on the actual microgeometry of the composite and it can be derived from theoretical considerations; can be the output of some numerical simulations; can also be the results of experimental measurements. In any way, the effective moduli so predicted would never violate the Hashin-Shtrikman bounds due to unilateral variation of the deviator. The numerical examples shown in this article clearly demonstrate this point.

References

[1] Torquato S. "Random heterogeneous media: microstructure and improved bounds on effective properties". *Applied Mechanics Review*, Vol. 44, pp 37-76, 1991.

[2] Kachanov M., Tsukrov I. and Shafiro B. "Effective moduli of solids with cavities of various shapes". *Applied Mechanics Review*, Vol. 47, pp S151-S174, 1994.

[3] Mori T., and Tanaka K. "Average stress in matrix and average elastic energy of materials with misfitting inclusions". *Acta. Met.*, Vol. 21, pp 571-574, 1973.

[4] Weng GJ. "The theoretical connection between Mori-Tanaka's theory and Hashin-Shtrikman-Wapole bounds". *Int. J. of Engineering Science*, Vol. 28, pp 1111-1120, 1990.

[5] Benveniste Y. "Some remarks on three micromechanical models in composite media". *Journal of Applied Mechanics*, Vol. 57, pp 474-476, 1990.

[6] Dunn ML and Taya M. "Micromechanics prediction of the effective electroelastic moduli of piezoelectric composites". *Int. J. Solids Struct.*, Vol. 30, pp 161-175, 1993.

[7] Christensen RM. "A critical evaluation for a class of micromechanics models". *J. Mech. Phys. Solids*, Vol. 38, pp 379-404, 1990.

[8] Wu TT "The effect of inclusion shape on the elastic moduli of a two-phase material". *Int. J. of Solids Struct.*, Vol. 2, pp 83-88, 1965

[9] Willis JR "Variational and related methods for the overall properties of composites". *Adv. Appl. Mech.* Vol.21, pp 1-78, 1981.

[10] Eshelby JD " The determination of the elastic field of an ellipsoidal inclusion, and related problems". *Proc. Royal. Soc. London* Vol. 241A, pp 376-396, 1957.

[11] Taya M and Chou TW. "On two kinds of ellipsoidal inhomogeneities in an infinite elastic body: an application to a hybrid composite". *Int. J. Solids Struct.* Vol. 17, pp 553-563, 1981.

[12] Hashin Z and Shtrikman S. "A variational approach to the theory of the elastic behavior of multiphase materials". *J Mech. Phys. Solids,* Vol.11, pp 127-140, 1963.

[13] Weng, GJ. "Some elastic properties of reinforced solids, with special reference to isotropic ones containing spherical inclusions", *Int. J. Engng. Sci.* Vol. 22, pp 845-856, 1984.

[14] Chen HS and Acrivos A. "The effective elastic moduli of composite materials containing spherical inclusions at non-dilute concentrations". *Int. J. of Solids Struct.*, Vol. 14, pp 349-364, 1978.

[15] Chiang CR. "A note on eigenstrains". *Philosophical Magazine.* Vol. 54A, ppL45-L47, 1986.

[16] Budiansky B. "On the elastic moduli of some heterogeneous materials". *J Mech. Phys. Solids*, Vol. 13, pp 223-227, 1965.

[17] Hill R. "Elastic properties of reinforced solids: some theoretical principles". *J Mech. Phys. Solids*, Vol. 11, pp 357-372, 1963.

[18] Batchelor GK. "Sedimentation in a dilute dispersion of spheres". *J. Fluid Mech.* Vol. 52, pp245-268, 1972.