SIZE EFFECT FACTORS ON THE GAUGE LENGTH DEPENDENCE OF CERAMIC MONOFILAMENT TENSILE STRENGTH

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Abstract

Weibull weakest link model has been widely applied for the fracture probability scaling of brittle materials. However, a discrepancy has been reported on some brittle monofilaments between the estimated fracture probability and the derived data in the way as if the Weibull parameters were dependent on the gauge length.

This paper discusses an interpretation of flaw size distribution for Weibull parameters on the strength size-effect of brittle monofilaments. A model on monofilament strength size-effect was firstly presented through coupling a Griffiths fracture mechanics and a distribution of crack initiating flaw size. The model parameters and Weibull parameters were secondly compared to provide a relationship between Weibull parameters and flaw size distribution. The results have read to a mathematical model of Weibull parameter dependence on the monofilament gauge length. The standard deviation of the flaw size distribution was found in the model an important factor of the gauge length dependence. Then thirdly the mathematical model was applied on the Weibull parameters of Tyranno ZMI Si-Zr-C-O monofilament strength to assess if real monofilament reveals through the modeling the parameter dependence on the gauge length. The result have indicated that the effectiveness of Weibull scaling might be questioned due to the variable Weibull parameters for the case to scale the fiber strength of 'pull-out length' level gauge using the data of rather long samples for monofilament tensile test.

1 Introduction

Fiber reinforced composites have been expanding the applications for aerospace lightweight structures due to the excellence in the specific strength. Although the distribution in the strength has challenged the material reliability, steady efforts for the database and the design expertise have enabled composite main structures such as carbon fiber reinforced plastic (CFRP) fuselages and wings for civil aircrafts. On civil aero engines, however, metallic materials have been still used for the main components such as rotor disks and torque shafts. The failure of aero engine main components may lead to a tragic loss of life and the investment for a new engine. Hence, the reliability of well-matured metallic materials still has both safety and economic benefits, even though composites may lead to a drastic weight reduction and automotive composite drive shaft and brake disk are already on the commercial market. Thereby the further improvement on the strength reliability is one of the key factors of composites for the lightweight aero engine component applications.

Based on the Weibull weakest-link model, theoretical efforts have been made for describing the strength and the reliability of composites[1]. However, some of the composite reinforcement fibers have been reported the discrepancies between the estimated fracture probabilities by the Weibull model and the derived experimental data. Hitchon and Phillips measured the carbon fiber strength of different gauge lengths to have shown that Weibull scaling did not accurately reproduce the strength distribution at a gauge length with the parameters of another gauge length[2]. Knoff showed in developing a modified weakest link model that classical weakest link model provided strength of fiber continue to increase at smaller test length even though real fibers frequently did not show the tendency[3]. Lavaste, Besson, and Bunsell also reported that classical Weibull distribution function could not account for the observed dispersions of different gauge lengths on alumina and alumina-zirconia single fiber strength[4]. Gurvich, Dibenedetto, and Pegoretti showed that Weibull parameters vary with the fiber length by analyzing the data obtained separately for populations of E-glass and carbon fibers[5]. Pickering and Murray reported on carbon fibers the inadequacies of classical Weibull model to extrapolate the strength at long gauge length down to short length[6].

The factors of discrepancies between the Weibull scaling and experimental data have been discussed such as 1) experimental factors: 1-a)the strain rate may be a factor of the experimental error to influence the gauge length effects[7], 1-b)fibers generally have to be extracted from a bundle to cause longer samples break prior to testing at higher probability than shorter samples thus the 'preselection' can cause the error[4], 1-c)the probabilistic effects of the clamp on fiber tensile testing at various gauge length, or 'clamp effects,' should be included in Weibull scaling[8], 2)fiber geometrical factors: geometrical irregularities along gauge length and between fibers at a bundle lead to the experimental error to hide the accuracy of Weibull scaling[9], and 3)modal factors: bimodal or multimodal distributions of flaw sizes may negate the accuracy of extrapolating the strength distribution data obtained at longer lengths down to fiber-matrix transfer lengths[10]. The authors have resolved the experimental factors 1) as 1-a)quasi-static tensile condition has been attained with a slow cross head speed capacity of Instron 5540 tensile test system, 1-b)fibers of a length have been preselected from a bundle to be sampled as tensile test specimen, and 1-c)a datum of sample, which fractured within 5mm from sample end, was not used for the further statistical analyses[11,12]. In addition, the authors have used elastic glue to minimize the stress concentration due to clamp or glue between sample fibers and the tabs. On the fiber geometrical factors 2), the authors have applied a laser scan micrometer (LSM, Mitutoyo Co. Ltd.), which is for fine wire assessment such as semiconductor bonding wire quality control, to measure sample fiber geometry in advance of the tensile test[11,12]. A 'representative diameter,' which is an imaginary uniform diameter, has been analytically derived for the strength calculations of variable diameter fibers[13]. The authors then coupled fiber tensile load and measured diameters to modify Weibull scaling for the material strength evaluation of variable diameter fibers. The results have revealed that Nicalon, Hi Nicalon, and Hi Nicalon Type S SiC fibers (Nihon Carbon Co. Ltd.) possess drastically higher potential strength, which is of imaginary uniform diameter fibers, than the known data[14]. The modal factors 3) have been assessed through intensive scanning electron microscope (SEM) analyses on tensile tested fiber fracture surfaces of several gauge length samples[11]. The results on Tyranno ZMI Si-Zr-C-O fibers (UBE Industry Ltd.) have shown that there were several kinds of crack-initiating flaws and the densities vary as the parameters of gauge length. Thus, single-modal Weibull model expected to provide a set of variable parameters by the sample gauge length.

In this paper, the authors further study the effect of flaw distribution on the parameters of single-modal Weibull model. A statistical interpretation is presented for the parameters on the strength size-effect of brittle monofilaments.

2 A Model of Critical Flaw Size on Variable Gauge Length Fiber

An imaginary uniform diameter brittle fiber is considered as schematically depicted in Fig.1: the fiber of gauge length 'L_0' contains flaws of distributed size 'ϕ.'

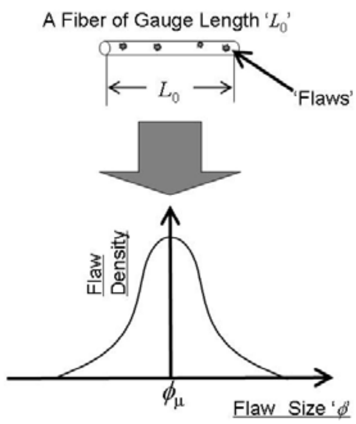

Figure 1 An imaginary model fiber

The probability density is assumed to be of the Gaussian distribution as follows on containing a flaw of the maximum size 'ϕ_0'.

$$f(\phi_0) = \frac{1}{\sqrt{2\pi}\Sigma_0} \cdot \exp\left\{ -\frac{1}{2}\left(\frac{\phi_0 - \phi_\mu}{\Sigma_0}\right)^2 \right\} \tag{1}$$

Where, 'Σ_0' denotes the standard deviation and 'ϕ_μ' the mean of flaw size. Another imaginary fiber is considered for the gauge length 'L_1,' where '$L_1 > L_0$.' At an expectation the flaw size 'ϕ_0' can shift to larger value by increasing the gauge length as 1)the gauge length is equivalent to the sample space of Gaussian distribution, 2)larger sample space provides higher expectation of the samples, and 3) a flaw of a size, which is larger than 'ϕ_0' and smaller in the expectation at the gauge length 'L_0,' can recover the expectation to the level of 'ϕ_0' flaw if larger gauge length is assumed. Thus, the maximum flaw size 'ϕ_1' may be approximated as follows between fibers of the length 'L_0' and longer gauge length 'L_1'.

$$\frac{L_1}{L_0\sqrt{2\pi}\Sigma_0} \cdot \exp\left\{ -\frac{1}{2}\left(\frac{\phi_1 - \phi_\mu}{\Sigma_0}\right)^2 \right\} = \frac{1}{\sqrt{2\pi}\Sigma_0} \cdot \exp\left\{ -\frac{1}{2}\left(\frac{\phi_0 - \phi_\mu}{\Sigma_0}\right)^2 \right\} \tag{2}$$

Figure 2 schematically depicts the relationship in Eq. (2).

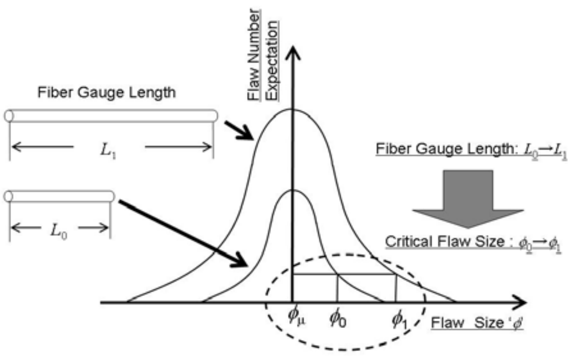

Figure 2 A schematics of flaw size - gauge length relationship

The Eq. (2) leads to a relationship as follows.

$$\phi_1 = \phi_\mu + \Sigma_0\sqrt{\left(\frac{\phi_0 - \phi_\mu}{\Sigma_0}\right)^2 + 2\ln\left(\frac{L_1}{L_0}\right)} \tag{3}$$

Figure 3 depicts the relationship (3) for the case C.V. $\Sigma_0/\phi_\mu = 0.1$ and the maximum flaw size 'ϕ_0' at 'L_0:' $\phi_0 = \phi_\mu + \Sigma_0$. The relationship (3) implies that larger flaws can exist in longer fibers, and the flaw size standard deviation 'Σ_0,' in addition to the mean size 'ϕ_μ,' is one of the key parameters of the flaw size variability.

Figure 3 Maximum flaw size – fiber gauge length curve for the case flaw size C.V. is 0.1 and the maximum flaw size is one sigma from the mean size

2.2. A Fracture Strength Size-Effect Model

Griffiths studied straight crack propagation in a flat homogeneous isotropic plate of uniform thickness being subjected to stresses[15]. The energy balance model has revealed a crack propagating stress in the form inversely proportional to the square root of the crack length. Many of the extensions such for 'penny-shape crack' have been also leaded to the form inversely proportional to the square root of flaw size on crack propagating stresses. Following approximation may thus provide a relationship of acceptable error level between the fiber fracture stress, 'σ_f,' and the size of critical crack nucleating flaw, 'ϕ.'

$$\sigma_f = \frac{Const}{\sqrt{\phi}} \tag{4}$$

The flaw size 'ϕ' must be the largest in the fiber, as the smaller flaws require higher stresses than 'σ_f' to nucleate critical cracks.

A relationship may be thus given between a fiber gauge length 'L' and the fracture stress 'σ_f' through the combination of (3) and (4).

$$\sigma_f = \frac{Const}{\sqrt{\phi_\mu + \Sigma_0 \sqrt{\left(\frac{\phi_0 - \phi_\mu}{\Sigma_0}\right)^2 + 2\ln\left(\frac{L}{L_0}\right)}}} \tag{5}$$

This relationship (5) assumes that 'ϕ_0' is the largest flaw size in the fiber of the gauge length 'L_0' and (3) estimates the largest flaw size for the fiber of the gauge length 'L.'

2.3. An Interpretation of Weibull Parameters through the Flaw Statistics

Single-modal Weibull weakest-link model appears as follows for the estimation of intrinsic mean strength through an extrapolation from a gauge length 'L_0' data to the 'L.'

$$\bar{\sigma} = \sigma_0 \left(\frac{L}{L_0}\right)^{-\frac{1}{m}} \Gamma\left(1 + \frac{1}{m}\right) \tag{6}$$

Where, 'Γ' is gamma function, '$\bar{\sigma}$' is a mean strength, and 'm and σ_0' are the Weibull shape and scale parameters at gauge length 'L_0,' respectively. The relationships (5) and (6) must remain within acceptable error level on the size-effect variable 'L/L_0.' Thus, first and second terms may be approximated equal between the Taylor series of Eq.(5) and Eq.(6) around '1' on 'L/L_0'. Thereby a relationship may be approximated as follows.

$$\left(\begin{array}{l}
\frac{Const}{\sqrt{\phi_0}} \\
-\frac{\Sigma_0^{\,2}}{2} \cdot \frac{Const}{\phi_0^{3/2} \cdot \left(\phi_0 - \phi_\mu\right)} \cdot \left(\frac{L}{L_0} - 1\right) \\
+\frac{Const}{\sqrt{\phi_0}} \left[\begin{array}{l} -\frac{\phi_0 - \phi_\mu}{2} \cdot \frac{\left\{\frac{-\Sigma_0^{\,2}}{2\left(\phi_0 - \phi_\mu\right)^2} - \frac{\Sigma_0^{\,4}}{2\left(\phi_0 - \phi_\mu\right)^4}\right\}}{\phi_0} \\ +\frac{3\Sigma_0^{\,4}}{8\phi_0^{\,2}\left(\phi_0 - \phi_\mu\right)^2} \end{array}\right] \\
\times \left(\frac{L}{L_0} - 1\right)^2 \\
+\cdots
\end{array}\right)$$

$$\cong \left(\begin{array}{l}
\sigma_0 \cdot \Gamma\left(1 + \frac{1}{m}\right) \\
-\frac{\sigma_0}{m} \cdot \Gamma\left(1 + \frac{1}{m}\right) \cdot \left(\frac{L}{L_0} - 1\right) \\
+\frac{1}{2} \cdot \frac{\sigma_0}{m} \cdot \left(1 + \frac{1}{m}\right) \cdot \Gamma\left(1 + \frac{1}{m}\right) \cdot \left(\frac{L}{L_0} - 1\right)^2 \\
+\cdots
\end{array}\right) \tag{7}$$

i.e.,

$$\begin{cases}
\frac{Const}{\sqrt{\phi_0}} = \sigma_0 \cdot \Gamma\left(1 + \frac{1}{m}\right) \\
\frac{\Sigma_0^{\,2}}{2} \cdot \frac{Const}{\phi_0^{3/2} \cdot \left(\phi_0 - \phi_\mu\right)} = \frac{\sigma_0}{m} \cdot \Gamma\left(1 + \frac{1}{m}\right)
\end{cases}. \tag{8}$$

$$\therefore \begin{cases}
m = \frac{2\phi_0\left(\phi_0 - \phi_\mu\right)}{\Sigma_0^{\,2}} \\
\sigma_0 = \frac{Const}{\sqrt{\phi_0} \cdot \Gamma\left(1 + \frac{\Sigma_0^{\,2}}{2\phi_0\left(\phi_0 - \phi_\mu\right)}\right)}
\end{cases} \tag{9}$$

The relationships in (9) have re-defined the Weibull parameters with the Gaussian distribution parameters of critical flaw size.

2.4. A model of variable Weibull parameters

The derived relationships in (9) have linked the parameters of Weibull distribution and Gaussian distribution. Thus, the expertise on Gaussian distribution can add Weibull parameters the aspects as the functions of gauge length. Fiber gauge length 'L_0' is equivalent to the sample size of Gaussian distribution on the flaw size statistics (1). Thus, the mean '$\phi_{\mu 1}$'and the standard deviation 'Σ_1' at a gauge length 'L_1' are given with the Gaussian distribution parameters at 'L_0' as follows.

$$\begin{cases} \phi_{\mu 1} = \phi_\mu \\ \Sigma_1 = \Sigma_0 \sqrt{\dfrac{L_1}{L_0}} \\ and\,(3) \\ \phi_1 = \phi_\mu + \Sigma_0 \sqrt{\left(\dfrac{\phi_0 - \phi_\mu}{\Sigma_0}\right)^2 + 2\ln\left(\dfrac{L_1}{L_0}\right)} \end{cases} \tag{10}$$

The relationships (10) may provide a set of Weibull parameters for the case 'L_0' is unknown thus one takes a gauge length 'L_1' as 'L_0,' by substituting the 'ϕ_1' and 'Σ_1' for the 'ϕ_0' and 'Σ_0' of Eq.(9). Figure 4 depicts the case of C.V. $\Sigma_0 / \phi_\mu = 0.1$ and the maximum flaw size 'ϕ_0' at 'L_0:' $\phi_0 = \phi_\mu + \Sigma_0$. A set of the parameters insensitive to 'L_1/L_0' is desirable for the Weibull scaling users to expect that Weibull parameters are independent of the fiber gauge length thus the data of various gauge length samples are equally used in the strength scaling. As depicted in Fig.4, however, the results showed a clear dependence of Weibull parameters on the fiber gauge length in the way the shape parameter 'm_1' increases and decreases while the scale parameter 'σ_1' decreases by assuming longer gauge length 'L_1' around the unit of gauge length'L_0.'

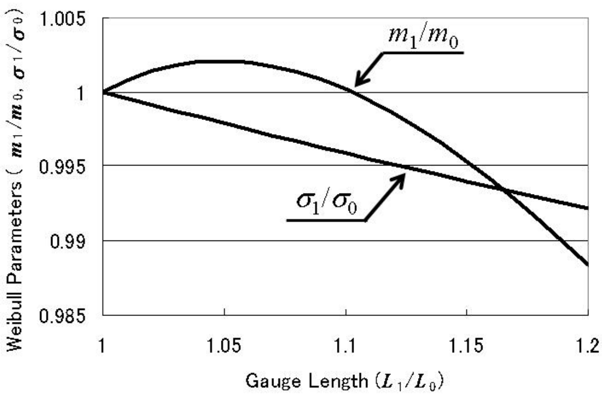

Figure 4 A schematics of Weibull parameter dependence on sample gauge length

This result may be important for the strength modeling of composites as many models assume gauge length insensitive Weibull parameters on the reinforcement fibers. What is more, it is noted that the case of small standard deviation 'Σ_0' leads to the 'gauge length insensitive' Weibull parameters as the small 'Σ_0' negates the factors of gauge length in Eq. (10).

3 Experiments

The mathematical model (9) and (10) were applied on the Weibull parameters of Tyranno ZMI Si-Zr-C-O monofilament strength to assess if real monofilament reveals through the modeling the parameter dependence on the gauge length.

Tyranno ZMI monofilaments of 20mm, 50mm, 100mm, 200mm gauges have been tensile tested with Instron 5542 tensile testing machine utilizing a 10N load cell, at the cross head speed of 0.1mm/min. Prior to the tensile tests, the diameters of randomly selected 30 samples of each gauge length were measured in 1 mm step along the gauge length using a laser scan micrometer LSM-500 with an accuracy of ±0.1 μm (Mitutoyo Corp., Kanagawa, Japan). Each sample was immersed in glycerin during the tensile test to recover the fragments. The fracture surfaces were then analyzed with scanning electron microscope (SEM, Hitachi Co. S-4700, Hitachi Japan).

4 Experimental Results and Discussion

Figure 5 is a typical SEM image of a fracture surface used in the measurement of crack nucleation source size and table I shows the derived parameters of fiber strength for the four different gauge lengths of 20mm, 50mm, 100mm, and 200mm.

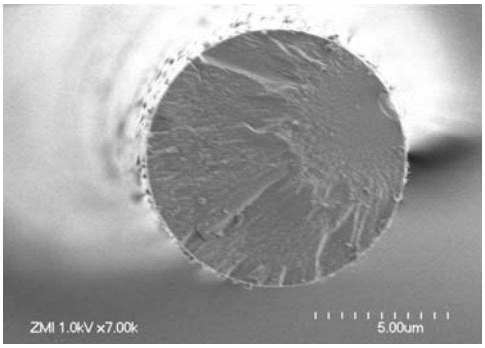

Figure 5 An Example of Tyranno ZMI Fracture surface

Table I Tensile Test Results on Tyranno ZMI

Gauge Length (*mm*)	Mean strength σ_m [GPa]	Standard deviation [×0.1GPa]
200	2.7	7.6
100	2.8	8.6
50	2.8	6.6
20	2.9	6.6

The SEM fractography had revealed that the crack nucleation points located mainly close to the fracture surface circumference, or fiber surface. The observed fracture surfaces had shown at the nucleation points particles or surface pre-cracks, which were surrounded by mirror area. The particles or pre-cracks were approximated to be elliptic in the size measurements and the equivalent size of defect, '*D*,' was defined as '$D=(\phi_l \cdot \phi_s)^{1/2}$', where '$\phi_l$' and '$\phi_s$' were longer and shorter axis, respectively. Figure 6 shows the derived relationship between the fracture stress 'σ_f,' which was calculated with the fracture portion diameter, and the equivalent size of defect '*D*,' with the error level of ±0.25μm.

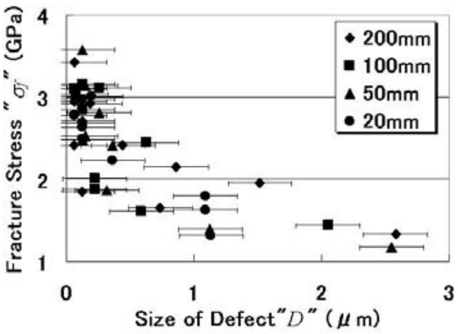

Figure 6 Fracture Stress against Size of Defect

Figure 7 is a modification of the Fig.6 from '*D*' to '$D^{-1/2}$' excluding the '*D*' of smaller than the error level of 0.25μm.

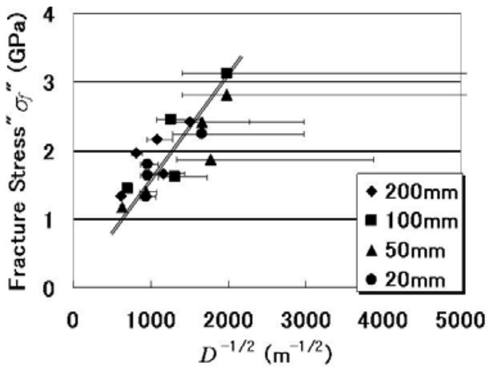

Figure 7 Fracture Stress against the Defect Size '*D*' in $D^{-1/2}$

The least-square fit in Fig.7 indicates that the fiber fracture stress, 'σ_f,' is approximately proportional to the '$1/D^{1/2}$' with the proportionality constant of 1.5 MPa·m$^{1/2}$, or $\sigma_f = 1.5/\sqrt{D}$ (MPa), as is expected on the Griffiths fracture mechanics. Thus, '*Const*' in Eq. (5) may be assumed 1.5 and following relationship may provide an acceptable approximation of Eq.(5) to fit the mean strength in Table I.

$$\sigma_f = \frac{1.5}{\sqrt{2.3\times10^{-7} + 4.8\times10^{-8}\cdot\sqrt{7.5\times10^{-1} + 2\ln(L/L_0)}}} \tag{11}$$

Where, 'L_0' is set 20mm. On this approximation the mean of flaw size 'ϕ_μ' is 2.3×10^{-1}μm, the standard deviation 'Σ_0' is 4.8×10^{-2}μm and the critical flaw size 'ϕ_0' is 2.7×10^{-1}μm. Thereby Eq.(9) provides a set of Weibull parameters for 20mm gauge length case as follows.

$$\begin{cases} m \cong 8.3 \\ \sigma_0 \cong 0.93(GPa) \end{cases} \tag{12}$$

These results read to a set of variable Weibull parameters as depicted in Fig.8, where 'L_0=20mm' is unknown thus one takes a gauge length 'L_1' as 'L_0,' by substituting the 'ϕ_1' and 'Σ_1' for the 'ϕ_0' and 'Σ_0' of Eq.(9).

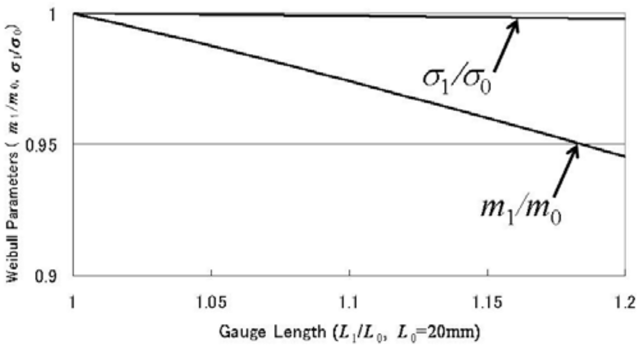

Figure 8 A derived Weibull parameter dependence on sample gauge length for Tyranno ZMI Si-Zr-C-O fiber

The Weibull shape parameter was found variable more than 5% by 20% level of gauge length variation. This result is important as Weibull scaling has been often applied on the 'pull-out' level length of fiber strength using monofilament tensile data: the variation in gauge length is far more than 20% level thus inadequate Weibull parameters might have been applied in composite strength modeling. Figure 9 shows the case of estimating weibull parameters with 20mm gauge tensile data down to 10mm gauge, using Eq.(9) and Eq.(10). The Weibull shape parameter was found slightly increases and drastically decreases by applying shorter gauge length of tensile samples. Thus, the authors deduce that it is not desirable for effective fiber strength scaling to apply a set of constant Weibull parameters of a gauge length to far different gauge length fibers.

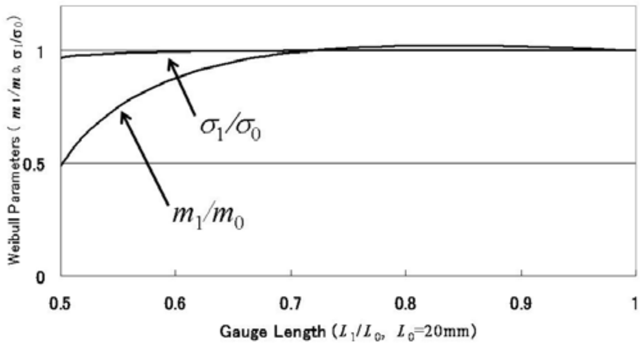

Figure 9 An estimation of Weibull parameters down to short gauge length of Tyranno ZMI Si-Zr-C-O fiber

5 Concluding Remarks

The authors have presented a model on monofilament strength size-effect through coupling a Griffiths fracture mechanics and a distribution of crack initiating flaw size. The model parameters were then compared with Weibull parameters to provide a relationship between Weibull parameters and flaw size distribution. Through the comparison a mathematical model was derived for the Weibull parameter dependence on the monofilament gauge length. The standard deviation of the flaw size distribution was found an important factor of the gauge length dependence of Weibull parameters. Thus, flaw size control for smaller standard deviation appeared to be a possible scheme of attaining a minimal size effect of ceramic fiber strength.

The gauge length dependence of Weibull parameters was assessed on Tyranno ZMI Si-Zr-C-O monofilament strength to see if real monofilament shows the parameter variability. The result have indicated that the effectiveness of Weibull scaling might be questioned for the case to scale the fiber strength of 'pull-out length' level gauge using the tensile test data of rather longer samples.

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