BOUNDS ON BUCKLING RESPONSE FOR ANISOTROPIC LAMINATED PLATES

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Abstract

Nondimensional parameters and equations governing the buckling behavior of rectangular symmetrically laminated plates are presented that can be used to represent the buckling resistance, for plates made of all known structural materials. Bounds for these nondimensional parameters are also presented that are based on thermodynamics. These bounds provide insight into potential gains in buckling resistance through laminate tailoring and composite-material development. As an illustration of this point, upper bounds on the buckling resistance of long rectangular orthotropic plates with simply supported edges and subjected to uniform axial compression and uniform shear are presented. The results indicate that the maximum gain in buckling resistance for tailored orthotropic laminates, with respect to the corresponding isotropic plate, is in the range of 33% for plates with simply supported edges ,under compression loading conditions. For the plates subjected to shear, the corresponding gains in buckling resistance are in the range of 90% and occurs for plates with flexural anisotropy

1 Introduction

Laminated composite materials lend themselves to elastic tailoring of anisotropic structural components - a feature that allows structural designers to customize the stiffness-critical response of structural elements such as flat plates and curved panels. The benefits of elastic tailoring may result in a reduction in structural weight or improved performance, which are very important to many widespread applications such as aircraft, spacecraft, and sporting goods. Typically, these benefits are obtained by simply ensuring that the laminate stiffnesses are different in the principal directions (an example of orthotropy), or by building in elements of anisotropy that couple response modes to obtain a desired effect (e.g., coupling of extension, contraction and inplane shear deformations).

This paper is concerned with identifying the limits of benefit of elastic tailoring in buckling driven design. It appears not to be appreciated that there is a definite upper bound to the potential for elastic tailoring. By presenting the buckling performance of the optimal lay-up with respect to the quasi-isotropic lay-up, the fractional performance gain by elastic tailoring is clearly shown. Furthermore, by expressing flexural stiffnesses in terms of material invariant stiffnesses and lamination parameters [1,2] it is possible to deduce theoretical upper bounds to buckling performance for all possible laminated composite materials. Such information should prove useful for developing new materials. As an illustration of this point, upper bounds on the buckling resistance of long rectangular anisotropic plates with simply supported edges and subjected to uniform axial compression or uniform shear loads are presented. The results indicate that the maximum gain in buckling resistance for tailored orthotropic laminates, with respect to the corresponding isotropic plate, is in the range of 36% for orthotropic plates with simply supported edges, irrespective of the loading conditions. For plates with flexural anisotropy, the corresponding gains in buckling resistance are in the range of 90% for plates subjected to compression.

2 Thermodynamic bounds for anisotropic plates

Mansfield [3] gives the expression for the strain energy, U_b, of a flat anisotropic plate subject to flexural loading as

$$U_b = \frac{1}{2}\iint_A \{\kappa\}^T \left[D^*\right]\{\kappa\} dxdy \qquad (1)$$

with

$$\{\kappa\}^T = \left\{ -\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, -2\frac{\partial^2 w}{\partial x \partial y} \right\}$$

$$[D] = \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{12}^* & D_{22}^* & D_{26}^* \\ D_{16}^* & D_{26}^* & D_{66}^* \end{bmatrix}$$

where **κ** are curvatures, given in terms of transverse deformation, w and plate coordinates x, y. The reduced stiffness [**D***] is given by

$$\left[D^*\right] = [D] - [B][A]^{-1}[B] \qquad (2)$$

where [**A**], [**B**] and [**D**] are the in-plane, coupling and flexural matrices, respectively.

Relations between D^*_{ij} are found by ensuring that the strain energy in Eq. (1) is positive for all loading. This is done by enforcing that the determinant of [**D***] is positive definite. This is conveniently done by first expressing them in terms of nondimensional parameters. To obtain a convenient nondimensional form of the total strain energy, the stiffness matrix [**D***] is nondimensionalized by the geometric stiffness $(D_{11}^* D_{22}^*)^{0.5}$ which has the advantage that it does not significantly vary with lamination [4],

$$\begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{12}^* & D_{22}^* & D_{26}^* \\ D_{16}^* & D_{26}^* & D_{66}^* \end{bmatrix} = \sqrt{D_{11}^* D_{22}^*} \begin{bmatrix} \alpha_B^2 & \nu_{Bf} & \alpha_B \gamma_B \\ \nu_{Bf} & \dfrac{1}{\alpha_B^2} & \dfrac{\delta_B}{\alpha_B} \\ \alpha_B \gamma_B & \dfrac{\delta_B}{\alpha_B} & \dfrac{\beta_B - \nu_{Bf}}{2} \end{bmatrix} \qquad (3)$$

with

$$\alpha_B = \sqrt[4]{\frac{D_{22}^*}{D_{11}^*}}; \quad \beta_B = \frac{\left(D_{12}^* + 2D_{66}^*\right)}{\left(D_{11}^* D_{22}^*\right)^{1/2}};$$
$$\gamma_B = \frac{D_{16}^*}{\left(D_{11}^{*\,3} D_{22}^*\right)^{1/4}}; \quad \delta_B = \frac{D_{26}^*}{\left(D_{22}^{*\,3} D_{11}^*\right)^{1/4}}; \qquad (4)$$

and where the additional nondimensional parameter ν_{Bf} is

$$\nu_{Bf} = \frac{D_{12}^*}{\sqrt{D_{11}^* D_{22}^*}} \qquad (5)$$

and represents a form of flexural Poisson's ratio.

Typically, in defining the conditions on the elastic material parameters, positive definiteness of the strain energy density is enforced, which is valid at every material point of a structure. Enforcing this condition on the integrand of Eq. (1) results in the requirement that the matrix defined by Eq. (3) be a positive-definite matrix, which yields relationships that α_B, β_B, δ_B, γ_B and ν_{Bf} must obey. Applying Sylvester's criteria for positive definiteness of a matrix yields the following requirements [5]:

$$\begin{gathered} \alpha_B > 0 \\ \left(1 - \nu_{Bf}^{\,2}\right) > 0 \\ \beta_B - \nu_{Bf} > 0 \\ \beta_B - \nu_{Bf} - 2\gamma_B^{\,2} > 0 \\ \beta_B - \nu_{Bf} - 2\delta_B^{\,2} > 0 \\ \left(\frac{\beta_B - \nu_{Bf}}{2} - \delta_B^{\,2}\right) \\ -\nu_{Bf}\left(\nu_{Bf}\left(\frac{\beta_B - \nu_{Bf}}{2}\right) - \delta_B \gamma_B\right) \\ + \gamma_B\left(\nu_{Bf}\delta_B - \gamma_B\right) > 0 \end{gathered} \qquad (6)$$

The second and third of these conditions give the following bounds on ν_{Bf} and β_B; that is,

$$-1 < \nu_{Bf} < 1 \text{ and } \beta_B > \text{-}1 . \qquad (7)$$

Because no apparent upper bound on β_B is given by Eqs. (6), bounds for γ_B and δ_B are also not apparent. As a result, in the present study, bounds on the nondimensional parameters are sought with respect to the buckling response of simply supported and clamped plates, not the material behaviour. For this class of problems, the buckling response is completely independent of ν_{Bf} and positiveness of the total strain energy is used, instead of positive definiteness of the strain energy density, to eliminate ν_{Bf} from consideration. Specifically, a modified form of the total strain energy is sought that is independent of ν_{Bf} and whose positiveness can be guaranteed by enforcing positive definiteness of the corresponding integrand. Thus, an alternate form of Eq. (1), the total strain energy of a plate, is used that produces structural-response bounds on the minimum number of nondimensional parameters required to characterise the buckling behavior of simply supported and clamped plates as follows.

The desired form of Eq. (1) is obtained by noting that it is possible to eliminate ν_{Bf} as a variable governing the structural response for several cases of practical interest in design; that is, plates for which the transverse buckling displacement $w = 0$ on the boundary. This simplification is done by integrating Eq. (1) by parts using Green's Theorem and enforcing $w = 0$ on the boundary of a finite-length plate or the periodic unit of an infinitely long plate to obtain

$$\iint \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} dxdy = \iint \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 dxdy \quad (8)$$

Using this expression, the strain energy components containing β_B and ν_{Bf} may be reduced to a single term in β_B ; that is,

$$\iint \left[2\nu_{Bf} \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(\beta_B - \nu_{Bf})\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] dxdy$$
$$= \iint \left[2\beta_B \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \right] dxdy \quad (9a)$$

or

$$\iint \left[2\nu_{Bf} \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(\beta_B - \nu_{Bf})\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] dxdy$$
$$= \iint \left[2\beta_B \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] dxdy \quad (9b)$$

which allows the total strain energy to be written as

$$\frac{U_b}{\sqrt{D_{11}^* D_{22}^*}} = \frac{1}{2}\iint_A \{\kappa\}^T \left[D_{\text{mod}}^*\right]\{\kappa\} dxdy \quad (10)$$

where $[D^*_{\text{mod}}]$ is a modified nondimensional stiffness matrix that is given by

$$\left[D_{\text{mod}}^*\right] = \begin{bmatrix} \alpha_B{}^2 & \beta_B & \alpha_B \gamma_B \\ \beta_B & \frac{1}{\alpha_B{}^2} & \frac{\delta_B}{\alpha_B} \\ \alpha_B \gamma_B & \frac{\delta_B}{\alpha_B} & 0 \end{bmatrix} \quad (11a)$$

or by

$$\left[D_{\text{mod}}^*\right] = \begin{bmatrix} \alpha_B{}^2 & 0 & \alpha_B \gamma_B \\ 0 & \frac{1}{\alpha_B{}^2} & \frac{\delta_B}{\alpha_B} \\ \alpha_B \gamma_B & \frac{\delta_B}{\alpha_B} & \frac{\beta_B}{2} \end{bmatrix} \quad (11b)$$

depending on whether coefficients of $\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$ or $\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$ are retained in the strain energy, respectively. Neither representation of $\left[D_{\text{mod}}^*\right]$ is satisfactory for the purpose of finding bounds by enforcing positive definiteness of the integrand, however, as the former has a value of zero for a leading diagonal term, which is impossible for the requirement of positive strain energy, whilst the latter does provide for the contribution of $\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$ in its representation of strain energy. It is more useful to represent the strain energy associated with β_B as a linear combination of the terms given by Eqs. (9); that is,

$$\iint \left[2\beta_B \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \right] dxdy = \iint \left[2\beta_B \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] dxdy$$
$$= \iint 2\beta_B \left[2m\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + n\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \right] dxdy \quad (12a)$$

which allows the strain energy to be represented as

$$\frac{U_b}{\sqrt{D_{11}^* D_{22}^*}} = \frac{1}{2}\iint_A \{\kappa\}^T \left[D_{\text{mod}1}^*\right]\{\kappa\} dxdy \quad (12b)$$

where $[D^*_{\text{mod}1}]$ is another modified nondimensional stiffness matrix that is given by

$$[D^*_{\mathrm{mod1}}]=\begin{bmatrix}\alpha^2 & n\beta & \alpha\gamma\\ n\beta & \dfrac{1}{\alpha^2} & \dfrac{\delta}{\alpha}\\ \alpha\gamma & \dfrac{\delta}{\alpha} & m\beta\end{bmatrix} \quad (13)$$

for which $2m+n=1$ must be satisfied in order for Eqs. (10) and (12b) to remain equivalent, where m and n are real-valued numbers. The modified stiffness matrix $[D^*_{\mathrm{mod1}}]$ is more general than $[D^*_{\mathrm{mod}}]$ because assumptions have not been made concerning the relative contribution of β to $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2}$ or $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2$.

A sufficient condition for positive-valued total strain energy of deformation is that the modified stiffness matrix $[D^*_{\mathrm{mod1}}]$ be positive definite. Applying Sylvester's criteria, once again, for positive definiteness of a matrix yields the following requirements:

$$\begin{gathered}\alpha_B>0\\ m\beta_B>0\\ (n\beta_B)^2<1\\ m\beta_B>\delta_B^2\\ m\beta_B>\gamma_B^2\\ -mn^2\beta_B^3+(2n\delta_B\gamma_B+m)\beta_B\\ -(\gamma_B^2+\delta_B^2)>0\end{gathered} \quad (14)$$

Combining the latter of these relationships with $2m+n=1$ results in the following cubic polynomial in $n\beta_B$

$$\frac{(n\beta_B)^3}{2}-\frac{\beta_B(n\beta_B)^2}{2}+\left(2\gamma_B\delta_B-\frac{1}{2}\right)(n\beta_B)+\left(\frac{\beta_B}{2}-\delta_B^2-\gamma_B^2\right)>0 \quad (15)$$

For any given values of δ_B and γ_B, Eq. (15) gives the minimum, value of β_B that corresponds to positive strain energy. Its dependency on n is of little consequence because the minimum value of β_B is determined directly by ensuring that the solution to Eq. (15) has three real-valued roots, which, in turn, is satisfied by ensuring that the discriminant of the third-order polynomial in Eq. (15) is zero; that is,

$$\frac{1}{27}\left(4\gamma_B\delta_B-\frac{\beta_B^2}{3}-1\right)^3+\frac{1}{4}\begin{pmatrix}2\gamma_B^2+2\delta_B^2-\frac{2}{3}\beta_B\\ -\frac{4}{3}\gamma_B\delta_B\beta_B+\frac{2}{27}\beta_B^3\end{pmatrix}^2=0 \quad (16)$$

which is independent of the parameters m and n. Simplification of Eq. (16) yields a fourth-order expression in β_B given by

$$\begin{aligned}&\frac{\beta_B^4}{27}-\frac{2}{27}\left(\gamma_B^2+\delta_B^2\right)\beta_B^3\\ &-\frac{1}{27}\left(2+20\gamma_B\delta_B-4\gamma_B^2\delta_B^2\right)\beta_B^2\\ &+\frac{1}{3}\left(4\gamma_B^3\delta_B+2\gamma_B^2+2\delta_B^2+4\delta_B^3\gamma_B\right)\beta_B\\ &+\begin{pmatrix}\frac{1}{27}-\gamma_B^4-\delta_B^4-\frac{2}{9}\gamma_B^2\delta_B^2-\frac{4}{9}\gamma_B\delta_B\\ -\frac{64}{27}\gamma_B^3\delta_B^3\end{pmatrix}=0\end{aligned} \quad (17)$$

Equation (17) is used herein to obtain the minimum value of β_B for given values of δ_B and γ_B. It is noted that for some values of δ_B and γ_B there are multiple solutions for β_B that satisfy Eq. (17). For these circumstances, the appropriate choice of the minimal β_B value is the one that also satisfies the thermodynamic conditions given in Eq.(14), and by so doing, provides a unique solution for β_B. Upon finding the minimal value for β_B, Eq. (15) is used to determine the value of the parameter n. It is useful to observe that Eq. (17) exhibits identical dependence on δ_B and γ_B, meaning that δ_B and γ_B have identical effects on the minimal value of β_B because they are interchangeable. The contours of minimal β_B , as given by Eq. (17), are depicted as a function of δ_B and γ_B in Fig. 1. It is noted that these contours ands indeed all relationships derived in Section 2 hold for symmetric laminates in which the [**B**] matrix is zero. In fact, Fig. 1 was first derived for symmetric laminates [4]. The advance, made here, is that the form of strain energy for the nonsymmetric plate is governed by the reduced stiffness matrix [**D***], as formulated by Mansfield [3] and used by authors such as Ashton [6] may be used to find thermodynamic bounds on the [**B**] matrix. Currently, we have bounds on [**D**] and [**D***]. By finding bounds on [**A**] matrix we can find by elimination , thermodynamic bounds on [**B**] matrix. Thermodynamic bounds on [**A**] matrix are found in the next section by considering its inverse, $[\mathbf{A}]^{-1}$.

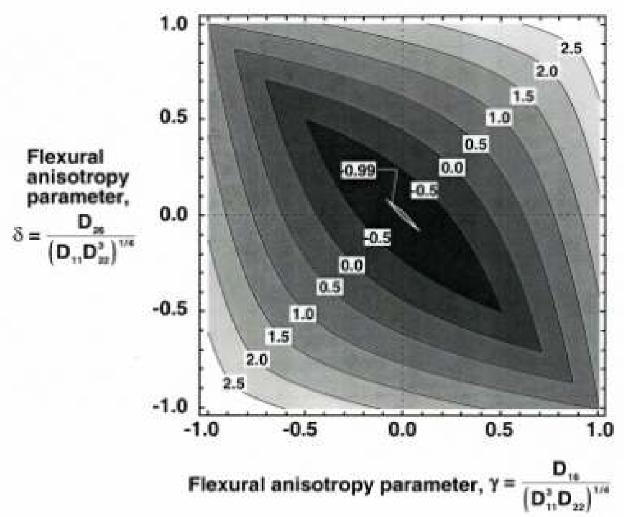

Fig. 1 Minimum value of β_B for given values of δ_B and γ_B.

3 Thermodynamic Bounds on [A] matrix

Thermodynamic bounds on the in-plane stiffnesses, [**A**] matrix, may be found by similar arguments to those made in Section 2. Mansfield [3] shows that the strain energy due to membrane loading is:

$$U_M = \frac{1}{2}\iint_A \{N\}^T [a]\{N\} dxdy \qquad (18)$$

with $[a] = [A]^{-1}$. Using a similar nondimensioning process done in Section 2, the strain energy may be written as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} = \sqrt{a_{11}^* a_{22}^*}\begin{bmatrix} \alpha_A^2 & \nu_{Af} & \alpha_A \gamma_B \\ \nu_{Af} & \frac{1}{\alpha_A^2} & \frac{\delta_A}{\alpha_A} \\ \alpha_A \gamma_A & \frac{\delta_A}{\alpha_A} & \frac{\beta_A - \nu_{Af}}{2} \end{bmatrix} \qquad (19)$$

with

$$\alpha_A = \sqrt[4]{\frac{a_{22}^*}{a_{11}^*}}; \quad \beta_A = \frac{\left(a_{12}^* + 2a_{66}^*\right)}{\left(a_{11}^* a_{22}^*\right)^{1/2}};$$
$$\gamma_A = \frac{a_{16}^*}{\left(a_{11}^{*\,3} a_{22}^*\right)^{1/4}}; \quad \delta_A = \frac{a_{26}^*}{\left(a_{22}^{*\,3} a_{11}^*\right)^{1/4}}; \qquad (20)$$

and where the additional nondimensional parameter ν_{Af} is

$$\nu_{Af} = \frac{a_{12}^*}{\sqrt{a_{11}^* a_{22}^*}} \qquad (21)$$

As done in Section 2, positiveness of the strain energy density, according to Sylvester's Theorem gives

$$\begin{aligned} &\alpha_A > 0 \\ &\left(1 - \nu_{Af}^2\right) > 0 \\ &\beta_A - \nu_{Af} > 0 \\ &\beta_A - \nu_{Af} - 2\gamma_A^2 > 0 \\ &\beta_A - \nu_{Af} - 2\delta_A^2 > 0 \\ &\left(\frac{\beta_A - \nu_{Af}}{2} - \delta_A^2\right) - \nu_{Af}\left(\nu_{Af}\left(\frac{\beta_A - \nu_{Af}}{2}\right) - \delta_A \gamma_A\right) \\ &+ \gamma_A\left(\nu_{Af}\delta_A - \gamma_A\right) > 0 \end{aligned} \qquad (22)$$

as thermodynamic constraints on material properties. The second and third of these conditions give the following bounds on ν_{Af} and β_A; that is,

$$-1 < \nu_{Af} < 1 \text{ and } \beta_A > -1. \qquad (23)$$

As done in Section 2 a fourth order polynomial in β_A is derived,

$$\begin{aligned} &\frac{\beta_A^4}{27} - \frac{2}{27}\left(\gamma_A^2 + \delta_A^2\right)\beta_A^3 \\ &- \frac{1}{27}\left(2 + 20\gamma_A\delta_A - 4\gamma_A^2\delta_A^2\right)\beta_A^2 \\ &+ \frac{1}{3}\left(4\gamma_A^3\delta_A + 2\gamma_A^2 + 2\delta_A^2 + 4\delta_A^3\gamma_A\right)\beta_A \\ &+ \left(\begin{aligned} &\frac{1}{27} - \gamma_A^4 - \delta_A^4 - \frac{2}{9}\gamma_A^2\delta_A^2 - \frac{4}{9}\gamma_A\delta_A \\ &- \frac{64}{27}\gamma_A^3\delta_A^3 \end{aligned}\right) = 0 \end{aligned} \qquad (24)$$

with constraints

$$\begin{aligned} \alpha_A &> 0 \\ m\beta_A &> 0 \\ (n\beta_A)^2 &< 1 \\ m\beta_A &> \delta_A^2 \\ m\beta_A &> \gamma_A^2 \\ -mn^2\beta_A^3 + (2n\delta_A\gamma_A + m)\beta_A & \\ -(\gamma_A^2 + \delta_A^2) &> 0 \end{aligned} \quad (25)$$

4 Thermodynamic Bounds on [B] matrix

We are now in a position to find the thermodynamic bounds on the [**B**] matrix. These bounds are implicit and are developed by combining Eqs. (2),(17) and (24) ensuring constraints (14) and (25) are satisfied. It is noted that constraints expressions pertinent for symmetric laminates, concerning the [**D**] matrix are found using Eqs. (14) and (17).

Further relations between D^*_{ij} are found from trigonometry using lamination parameters.

5. Buckling Performance

The buckling performance of long rectangular plates with simply supported edges under either compression or shear loading is considered. Nonsymmetric plates clearly reduce the effective bending stiffness as shown in Eq. (2) and reported by Ashton [6]. As such, only plates with flexural anisotropy are considered henceforth.

Weaver [7] gave an approximate expression for buckling coefficient, K_x, of a flexurally anisotropic plate, with simply supported edges, subject to compression loading as:

$$\begin{aligned} K_x &= \frac{N_x b^2}{\pi^2 (D_{11}D_{22})^{1/2}} \\ &= 2(1+\beta) - 2(\beta + 3 + 2\gamma^2)\frac{(\gamma+3\delta)^2}{(\beta+3)^2} \\ &\quad - 4(\delta + 2\gamma^3 - \beta\gamma)\frac{(\gamma+3\delta)^3}{(\beta+3)^3} \end{aligned} \quad (26)$$

where N_x is the critical value of the compression load at the point of buckling. Note, that the subscripts have been dropped from the nondimensional parameters in Eq. (26) and reflect symmetrical variants of those defined in Eq. (4). It is noted that for orthotropic plates, Eq. (26) simplifies to

$$K_x = \frac{b^2 (N_x)^{cr}}{\pi^2 \sqrt{D_{11}D_{22}}} = 2(1+\beta) \quad (27)$$

which provides $K_x = 4$ for isotropic materials (with $\beta = 1$). From Eq. (26) it is observed that flexural anisotropy, via Eq. (4) reduces buckling loads and confirm earlier work [8,9,].

Following Reference [4] the maximum buckling load, normalized with respect to the quasi-isotropic laminate is

$$\begin{aligned} \frac{N_x}{N_{x(\text{iso})}} &= \frac{(D_{11}D_{22})^{1/2}}{D_{iso}} \frac{K_x}{K_{x(\text{iso})}} \\ &= \varepsilon \frac{(1+\beta)}{2} \end{aligned} \quad (28)$$

where

$$\varepsilon = \frac{(D_{11}D_{22})^{1/2}}{D_{iso}} \quad (29)$$

.and D_{iso} is the quasi-isotropic flexural stiffness.
Reference [4] gives bounding values for the nondimensional parameters in Eq .(26) as

$$\begin{aligned} -1 < \beta < 3 \\ |\delta, \gamma| < 1 \end{aligned} \quad (30)$$

noting that when $\beta = 3$, $\varepsilon = 2/3$ [4]. Then the maximum value for buckling load is found to be

$$\left(\frac{N_x}{N_{x\,\text{iso}}}\right)_{\max} = \frac{4}{3} \quad (31)$$

for materials where E_{11}/E_{22} tends to infinity (where E_{11}, E_{22} are Young's moduli of a single ply). This result shows that there is an upper limit to tailoring lay-ups. It also shows that there is limited potential for elastic tailoring and the optimal lay-up provides only 33% better performance than the quasi-isotropic lay-up.

Whilst it is known that the presence of flexural anisotropy reduces buckling loads under compression loading (e.g. [7]) its effect is different for shear loading as it either raises or lowers buckling loads depending on the sign of shear (e.g. References [9] and [10]). An approximate formula for the shear buckling coefficient, K_{xy}, that includes the effect of flexural anisotropy, developed recently [11]

$$K_{xy} = \frac{N_{xy} b^2}{\pi^2 \left(D_{11} D_{22}{}^3\right)^{1/4}} = 3.42 + 2.05\beta - 0.13\beta^2 - 1.79\gamma$$
$$- 6.89\delta + 0.36\beta(2\gamma + \delta) - 0.25(2\gamma + \delta)^2 \quad (32)$$

quantifies matters. It is noted that there is a strong linear dependency on γ and even more so, on δ as reflected in the value of their coefficients. Noting that γ and δ may be positively or negatively valued, provides the possibility of raising or lowering buckling loads depending on the sign of shear loading and the sign of δ and γ. For example, if positive shear is applied to the plate then an increase in performance may be gained by using laminates containing significant amounts of negative flexural anisotropy, i.e. negative δ and γ values. For negative shear the converse holds, i.e. laminates with positive flexural anisotropy raise buckling loads. Finally, it is of merit to emphasise that changing the sign of flexural anisotropy of the laminate is done by reversing the sign of the fibre angle for each layer in the laminate.

The approximate nature of Eq. (32) is such that it is conservative over the entire range of nondimensional parameters and is observed to under predict buckling loads by approximately 10% in the vicinity of maximum buckling loads. As such, approximate bounds of performance are found by using a near-optimal lay-up in Eq. (32). Previously [8], it was found that the orthotropic lay-up containing the fibre angle 60^o was optimal for shear of a long plate with simply supported edges. To make use of flexural anisotropy, a ply orientation of –60^o gives large negative δ values and near maximum buckling loads. The approximate upper bound performance for shear buckling of a plate with infinitely large values of orthotropy ratios, $\frac{Q_{11}}{Q_{22}}$ and $\frac{Q_{11}}{Q_{12} + 2Q_{66}}$ is found by using a value of ply orientation of –60^o into Eq. (14) for the non-dimensional parameters. Then,

$$\frac{N_{xy}}{N_{xy_{iso}}} \approx 1.9 \quad (33)$$

where accuracy has been retained to 1 decimal place to reflect the approximate nature of Eq. (32). Of significance is the greater scope for anisotropic elastic tailoring using flexural anisotropy. Incorporating flexural anisotropy increases the upper bound on buckling load from 136% [8] to 190% of the quasi-isotropic lay-up, which is approximately a 50% increase.

6. Conclusions

Nondimensional parameters and equations governing the buckling behavior of rectangular symmetrically laminated plates have been presented. These nondimensional parameters can be used to represent the buckling resistance of rectangular plates, made of all known linearly elastic structural materials, in a very general, insightful, and encompassing manner. In addition, these parameters can be used to assess the degree of plate orthotropy, to assess the importance of anisotropy that couples bending and twisting deformations, and to characterize quasi-isotropic laminates quantitatively. Bounds for these nondimensional parameters have also been presented that are based on thermodynamics. Knowing these bounds provides insight into potential gains in buckling resistance through laminate tailoring and composite-material development. As an illustration of this point, some of the bounds presented herein have been used to determine upper bounds on the buckling resistance of long rectangular orthotropic plates with simply supported subjected to uniform axial compression and uniform shear. The results indicate that the maximum gain in buckling resistance for orthotropic plates, with respect to the corresponding isotropic plate, through laminate tailoring is in the range of 33% for plates with simply supported edges under compression loading. For shear, flexurally anisotropic plates are found to be better performing than orthotropic ones. It was shown that it is possible to increase buckling performance by 50% by using flexural anisotropy.

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