DESIGN OF COMPOSITE PLATE WITH OPTIMALLY DISTRIBUTED SHORT FIBERS

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Abstract

A new design method of functional fiber reinforced plastics (FRP) material which imitates the micro structure of bone or shell in the natural world is proposed in this work. The material has local anisotropy or variable stiffness property induced by distributed short fibers.

The present design method combines FEM with a lay-up design process the authors proposed recently. The first step of optimization is to calculate the optimum lamination parameter distribution by a gradient method. Then, the repetition of the design process in each element finds optimum short fibers distribution. The maximum deflection of thin plates is chosen as an objective function to be minimized in the optimization and it turned out that the variable stiffness plates always give lower deflection compared to conventional plates which reinforced by parallel fibers. Moreover, it is also revealed that optimally distributed fibers tend to be allocated with certain tendency.

1 Introduction

Some materials in the natural world have local anisotropy or variable stiffness. For example, an internal structure of bone is fabricated by a spongy structure composed of optimally distributing voids, which enable it to keep both enough strength and lightness. In recent literature, such structure which is allowed to vary the stiffness property continuously is named as variable stiffness [1] or locally anisotropic plates. The concept of variable stiffness is employed to a design of fibrous laminated composite plates in the present work.

Conventional laminated composite plates are fabricated typically by stacking orthotropic plies, each of which is composed of reinforcing straight fibers and matrix materials. It is known that structural designers can make use of the fiber orientation angles in the plies to design the overall mechanical properties most effectively. For years, tailoring of such composite plates has been done by varying the orientation of parallel fibers or thickness of the plies. Recent development in manufacturing techniques makes it possible to fabricate composite materials with fiber orientations that vary continuously. This allows us to make the composite plate reinforced by curvilinear fibers and to distribute the stiffness property more flexibly in the plate. Also noted is that the design space for fibrous composite can be dramatically expanded when it is compared to the conventional stacking sequence design problem. Thus, possibility is now available to improve the plate property significantly.

Martin and Leissa [2] presented a variable stiffness concept to improve the buckling performance of the plate using the Ritz method. Hyer and Lee [3] used the finite element method with curvilinear fibers to analyze strength and buckling performance of variable stiffness plates. They varied the fiber orientation angle from one element to another, and it turned out that such plates have higher failure load than that of plates with parallel fibers. Gürdal and Olmendo [1] proposed an analysis method of the in-plane response of a variable stiffness panel reinforced by sinusoidal wave fibers using a system of coupled elliptic partial differential equations. An improvement in buckling performance of plate by varying shape of fibers was successfully confirmed. Setoodeh and his co-workers [4, 5] combined the gradient method with the finite element method. They calculated optimum lamination parameter distribution that yields lower compliance and better buckling performance. But they did not obtain actual curvilinear fiber path from the obtained parameters.

The motivation of this study is to find optimum curvilinear fiber path from lamination parameter distribution. As an initial approach of such problem, an optimum short fiber distribution is determined for minimizing the maximum deflection in this paper. An angle deciding method from lamination parameters proposed by authors [6] is locally applied to each element of the finite element method to decide fiber orientation angles in the element. It is revealed that the plates with optimally distributed short fibers have higher stiffnesses than the conventional plates. It is also found that the short fibers tend to allocate with a certain directional tendency, and this suggests a possibility that an optimum continuous fiber distribution may exist.

2 Optimization Procedures

2.1 Lamination Parameters and Bending Stiffness

The laminated plates considered here have symmetrically laminated $2K$ plies. Due to the middle surface symmetry, there is no coupling between bending and extension. Thus, the bending of plates is characterized only by bending stiffness. The differential equation governing bending of symmetrically laminated plates, as shown in Fig. 1, is

$$\begin{aligned} & D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} \\ & + D_{22}\frac{\partial^4 w}{\partial y^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} \\ & = P(x,y) \end{aligned} \tag{1}$$

where w denotes the deflection of the plate. The bending stiffnesses D_{ij} $(i, j = 1, 2, 6)$ are

$$\begin{Bmatrix} D_{11} \\ D_{22} \\ D_{12} \\ D_{66} \\ D_{16} \\ D_{26} \end{Bmatrix} = \begin{bmatrix} U_1 & W_1 & W_2 \\ U_1 & -W_1 & W_2 \\ U_4 & 0 & -W_2 \\ U_5 & 0 & -W_2 \\ 0 & W_3/2 & W_4 \\ 0 & W_3/2 & -W_4 \end{bmatrix} \begin{Bmatrix} 1 \\ U_2 \\ U_3 \end{Bmatrix} \tag{2}$$

where U_i $(i = 1, 2, 3, 4, 5)$ are material invariants. The lamination parameters W_i $(i = 1, 2, 3, 4)$ that are made non-dimensional by $\zeta_k = z_k/(h/2)$ are written as

$$\begin{aligned} W_1 &= \sum_{k=1}^{K}(\zeta_k^3 - \zeta_{k-1}^3)\cos 2\theta_k, \\ W_2 &= \sum_{k=1}^{K}(\zeta_k^3 - \zeta_{k-1}^3)\cos 4\theta_k, \\ W_3 &= \sum_{k=1}^{K}(\zeta_k^3 - \zeta_{k-1}^3)\sin 2\theta_k, \\ W_4 &= \sum_{k=1}^{K}(\zeta_k^3 - \zeta_{k-1}^3)\sin 4\theta_k \end{aligned} \tag{3}$$

where θ_k are kth ply's fiber orientation angles. As given in Eqs. 3., the lamination parameters are defined for whole thickness of the plates. Thus the number of design variables is four for each element and it is independent of the number of plies. This feature is advantageous from a view point of optimization. The parameters depend on each other because of their trigonometric relations. The relationships among the parameters are

$$\begin{aligned} & W_1^2 + W_3^2 \le 1 \\ & \left(W_2 - W_1^2 + W_3^2\right)^2 + \left(W_4 - 2W_1W_3\right)^2 \\ & \qquad \le \left(1 - W_1^2 - W_3^2\right)^2 \end{aligned} \tag{4}$$

Equations 4 are used as constraints in the parameter optimization, and the parameters form a convex feasible region. This is also advantageous for optimization.

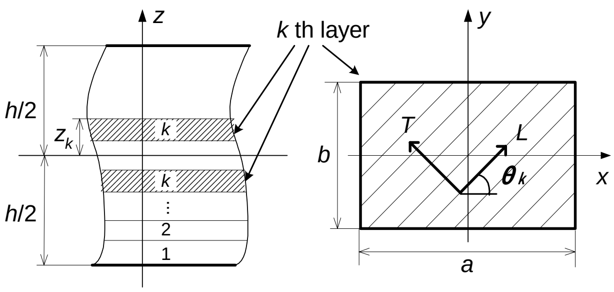

Fig. 1. Symmetric $2K$-ply laminated composite plate

2.2 Lamination Parameter Optimization

To determine the short fiber angles, the optimum distribution of lamination parameters is calculated first. Each element in the finite element method has a set of lamination parameters as independent design variables in the stiffnesses $[D_{ij}^{(n)}]$ $(i, j = 1, 2, 6;\ n = 1, 2, \ldots, ne)$, where ne is the total number of elements.

The optimization problem for the lamination parameters is formulated as

Minimize $w_{\max}(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, W_4^{(n)})$

such that $W_1^{(n)2} + W_3^{(n)2} \le 1$

and

$$\left(W_2^{(n)} - W_1^{(n)2} + W_3^{(n)2}\right)^2 + \left(W_4^{(n)} - 2W_1^{(n)}W_3^{(n)}\right)^2 \le \left(1 - W_1^{(n)2} - W_3^{(n)2}\right)^2$$

$(n = 1, 2, \ldots, ne)$

where $w_{\max}$ is the maximum deflection of plate and $W_i^{(n)}$ are lamination parameters for element n. The objective function for the maximum static deflection is defined by

$$w_{\max}^{*} = \frac{100 D_0}{P_0 a^2} w_{\max} \tag{5}$$

where $D_0 = E_T h^3 / 12(1-\nu_{LT}\nu_{TL})$ is a reference stiffness, and P_0 is representative load in the out-of-plane direction. As an optimizer, the modified feasible direction method was adopted in the ADS program [7] with the golden section method in one-dimensional search, as referred in Fukunaga et al [8].

2.3 Angle Deciding Process

Optimum fiber orientation angles are determined from lamination parameters by minimizing the error

$$E = \left(W_1 - \overline{W}_1\right)^2 + \left(W_2 - \overline{W}_2\right)^2 + \left(W_3 - \overline{W}_3\right)^2 + \left(W_4 - \overline{W}_4\right)^2 \tag{6}$$

between the optimum parameters and discrete parameters $\overline{W}_i$ (i = 1, 2, 3, and 4) for all possible discrete lay-ups [6]. Although a vast number of discrete lay-ups exit (i.e. if one uses a five-degree increment angle, the total number of lay-ups is $36^4 \approx$ 1.68 millions), Eqs. 3 take simple forms and all the parameters can be calculated easily. Thus, repetition of this procedure in each element gives the optimum short fibers distribution. The accuracy of the angle deciding process has been confirmed in the literature [9].

2.4 Finite Element Formulation Modified in the Present method

For symmetrically laminated thin plates, the strain energy stored in the plate during bending is given by

$$U = \frac{1}{2}\iint_A \{\kappa\}^T [D] \{\kappa\} dA \tag{7}$$

where

$$\{\kappa\} = \left\{ -\frac{\partial^2 w}{\partial x^2} \quad -\frac{\partial^2 w}{\partial y^2} \quad -2\frac{\partial^2 w}{\partial x \partial y} \right\}^{\mathrm{T}} \tag{8}$$

is a vector of bending curvatures, consisting of second derivatives of w, and $[D]$ is the bending stiffness coefficient matrix

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \tag{9}$$

that relates the moment resultant

$$\{M\} = \{M_x \quad M_y \quad M_{xy}\}^{\mathrm{T}} \tag{10}$$

to the curvature Eq. 8. by the relationship

$$\{M\} = [D]\{\kappa\}. \tag{11}$$

In applying the present optimization, each element has different stiffness $[D_{ij}^{(n)}]$ due to independent design variables $W_i^{(n)}$.

Suppose a rectangular finite element with nodes i, j, k, l and the deflection in the element is expressed [10] by

$$w(x, y) = [N]\{d_e\} \tag{12}$$

where $\{d_e\}$ is the element displacement vector

$$\{d_e\} = \{d_i \quad d_j \quad d_k \quad d_l\} \tag{13}$$

obtained by listing four nodal displacement vector such as

$$\{d_i\} = \left\{ w_i \quad \frac{\partial w_i}{\partial x} \quad \frac{\partial w_i}{\partial y} \right\}. \tag{14}$$

The shape function $[N]$ is written as

$$[N] = \{P\}[C]^{-1} \tag{15}$$

where $\{P\}$ and [C] are defined by using

$$w(x, y) = \{P\}\{\alpha\} \tag{16}$$

$$\{P\}=\{1 \quad x \quad y \quad x^2 \quad xy \quad y^2 \quad x^3 \quad x^2y \quad xy^2 \quad y^3 \quad x^3y \quad xy^3\} \tag{17}$$

$$\{\alpha\}=\{\alpha_1,\alpha_2,\alpha_3,\cdots,\alpha_{12}\} \tag{18}$$

and

$$\{d_e\}=[C]\{\alpha\}. \tag{19}$$

The curvature vector is obtained by

$$\{\kappa\}=[Q]\{\alpha\}=[Q][C]^{-1}\{d_e\} \tag{20}$$

where [Q] is derived from Eqs. 8, 16, 17 and 18. Substitution of Eq. 20 into Eq. 7 yields the strain energy

$$U_e=\frac{1}{2}\{d_e\}^T[K_e]\{d_e\} \tag{21}$$

written in terms of the element displacement vector, where

$$[K_e]= ([C]^{-1})^T\times\iint_A [Q]^T[D][Q]dA\times[C]^{-1} \tag{22}$$

in the element stiffness matrix in the present formulation.

Following similar procedure, the work due to transversely distributed load q is written by

$$W=-q(x,y)w(x,y). \tag{23}$$

The contribution of the external forces at nodes due to distributed load can be assigned by inspection. As in the standard finite element procedure, one obtains the global static equation as

$$[K]\{d\}=\{f\}. \tag{24}$$

where [K], {d} and {f} are the global stiffness matrix, global displacement vector and global load vector, respectively.

3 Numerical Results and Discussion

3.1 Optimum Lamination Parameter Distribution

Numerical results are given for symmetric 8-ply laminated square plates, and the elastic constants used for calculation are taken for a graphite/epoxy composite:

E_L = 138 [GPa], E_T = 8.96 [GPa], G_{LT} = 7.1 [GPa], and ν_{LT} = 0.30.

The boundary conditions for plates are designated by letters F, S and C in the direction of counterclockwise starting from the plate left edge, where F, S and C stand for free, simply supported and clamped edges, respectively.

Figure 2 presents the optimum lamination parameter distribution calculated by the present method for symmetrically laminated square plate with all edges clamped (CCCC) under uniformly distributed load. Note that Figs. 2 are not a result for some plate with specific number of plies. Instead, they represent results for entire plate because lamination parameters are defined for whole thickness of the plate (See section 2.1). Also note that the definitions of W_2 and W_3 in this study are inverse to literature [4]. The number of finite elements employed is 10 × 10, although 20 × 20 elements are used in the literature [4]. The calculated distribution in the present study corresponds well with that of the literature. Therefore, the validity of the lamination parameter optimization method of the present study is confirmed.

3.2 Optimum Short Fiber Distribution

The aim of this study is to determine optimum curvilinear fiber path of fibrous composite laminated plates. However, as the initial approach of such

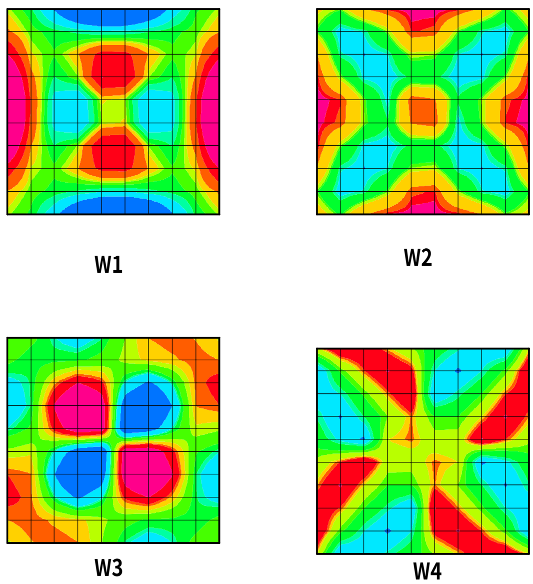

Fig. 2. Optimum distribution of lamination parameters for symmetrically laminated square plate with all edges clamped (CCCC) under uniformly distributed load (10 × 10 elements)

study, the optimum short fiber distribution is determined in this study. Applying the angle deciding method explained in section 2.3 to all elements in the finite element method sequentially yields the optimum short fiber distribution. The results for each ply which is calculated from the optimum lamination parameter distribution (Fig. 2.) are shown in Fig. 3. The number of symmetric plies is eight.

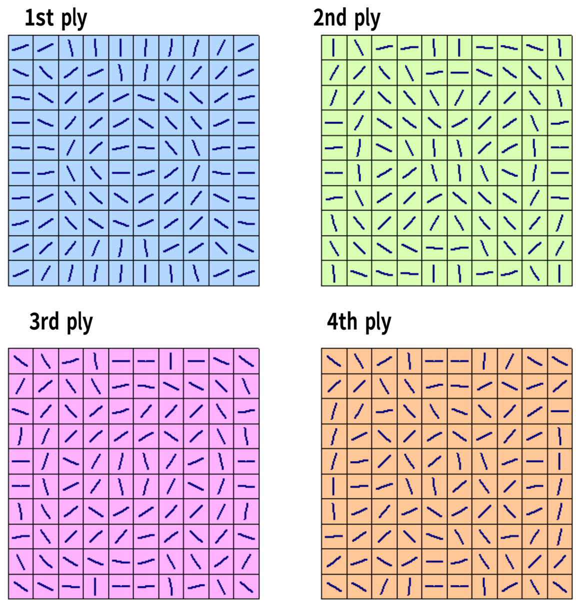

Fig. 3. Optimum short fiber distribution in each ply for symmetric 8-ply square plate with all edges clamped under uniformly distributed load (10 × 10 elements)

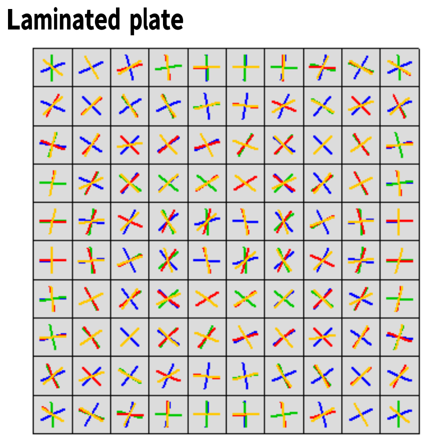

Fig. 4. Overlapped view of optimum short fiber distribution for symmetric 8-ply square plate with all edges clamped under uniformly distributed load (10 × 10 elements)

The square painted in blue shows the first (outermost) ply. The green, pink, and orange square present the second, third, and fourth ply, respectively. One can see clear tendency in the first ply. The fibers in the outer two elements adjacent to the boundary radiate toward to the center of the plate, and the fibers in the inner ones are allocated concentrically. On the other hand, in the second ply, the fibers in the outer two elements are arranged concentrically, and the fibers in the inner ones radiate. However, the tendency of the fibers is weak compared to the first ply. It is weakened gradually as going to inner plies, and no concrete tendency in the orientation is found at the fourth layer.

To confirm the effectiveness of the fourth layer, the overlapped view of Fig. 3 is shown in Fig. 4. Each color line corresponds with each ply's fibers in Fig. 3, that is, blue lines represent the fibers in the first ply. In a similar way, green, red, and orange lines represent the second, third, and fourth plies' fibers, respectively. The overlapped view gives much clearer tendency for concentricity and reveals inner lines' contribution to forming a clear shape. The fibers in the fourth ply are allocated randomly at first sight, however the orange lines don't turn to wrong way in the overlapped view. Therefore, it is found that the fiber distribution of inner ply is not negligible, either.

3.3 Comparison with Conventional Plates

To study the stiffness of the variable stiffness plates with optimally distributed short fibers, comparison is made in terms of the maximum deflection between the result for present plates and the result for the conventional plates which are composed of parallel fibers.

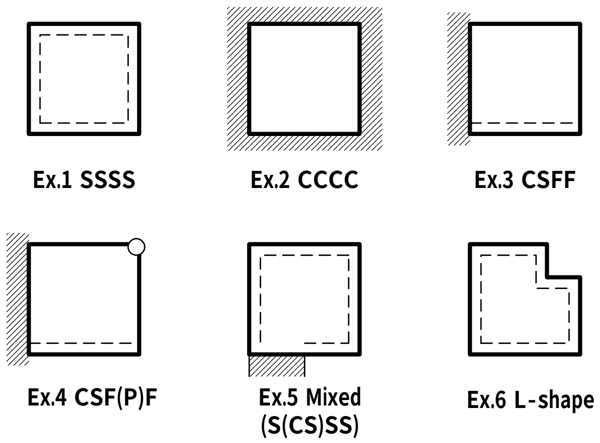

Fig.5. Boundary condition examples used in the comparison with the conventional plates

Six examples of boundary conditions are employed for comparison and are shown in Fig. 5. All example plates except Ex. 6 are square and they have length a. Ex. 1 is totally simply supported (SSSS) plate. Ex. 2 is totally clamped (CCCC) plate. Ex. 3 is the plate with asymmetrical boundary condition (CSFF). Ex. 4 is the plate with a point support at the free corner of Ex. 3. Ex. 5 has mixed boundary conditions. A half of a lower edge ($0.5a$) is clamped and others are simply supported. Ex. 6 is the plate with notch whose length is $0.2a$ and all edges are simply supported. Numerical results are given for all examples when the plates are under concentrated load at the center of the plate.

Table 1. Differences of the minimized maximum deflections for the variable stiffness plate and the conventional plates

	Variable Stiffness	LO+FEM (10x10) optimum lay-ups	Difference
Ex1	0.144	0.149 [45/-45/-45/-45]s	-3.47%
EX2	0.0664	0.0876 [0/90/90/-65]s	-31.9%
EX3	0.529	0.572 [15/-45/-45/-45]s	-8.13%
EX4	0.179	0.227 [45/-45/-45/0]s	-26.8%
EX5	0.119	0.126 [55/-50/-45/45]s	-5.88%
EX6	0.134	0.143 [45/-45/-45/45]s	-6.72%

Table 1 lists the minimized maximum deflections for the variable stiffness plates and the conventional plates, the optimum lay-ups for the conventional plates and differences of the deflection. The minimum values and optimum lay-ups for the conventional plates that were determined by Narita [11] are employed for comparison. Note that the maximum deflections for the conventional plate are re-calculated for 10 × 10 meshes because 20 × 20 meshes are used in the literature [11]. Since the present results give lower maximum deflections than the conventional plates for all the cases, it is uncovered that the variable stiffness plates can be designed to have higher stiffnesses than the conventional plates.

The overlapped views of the variable stiffness plates for examples in Fig. 5 are shown in Fig. 6. The results are also presented for the symmetric 8-ply plates. In the Ex. 1, cross lines through the center of plate can be seen. However, most of other elements have the angles of 45° and -45°. Those configurations are similar to those of the conventional plates, in other words, Ex. 1 plate has few differences in their structure when it is compared to the optimum conventional plate. Thus, the improvement is not significant (-3.47 %) and stays in the sixth among the examples. On the other hand, Ex. 2 has the highest improvement (-32.9 %), which may be caused by the tendency of the optimum short fiber distribution being much different from that of the conventional plate. The shape of Ex. 2 is clear and looks like “spider’s web”, but the optimum lay-up of the conventional plate ($[0/90/90/-65]_s$) is far from it. Therefore, the much difference in the improving value may be occurred. This is also said with respect to Ex. 4.

It is also revealed in Ex. 3 that the variable stiffness plates with optimally distributed short fibers have a certain tendency in their direction. In Ex. 3, the maximum deflection point is the upper right corner and the fibers in the first ply tend to flow from clamped left edge to maximum deflection point. Moreover, fibers in the inner three plies are allocated in the perpendicular direction to those in the first ply. Ex. 4 plate has S-shape orientation tendency. Ex. 5 and Ex. 6 give similar ones with Ex.1, but fibers near the clamped edge tend to curve in Ex. 5. As discussed above, it is clearly seen that the optimum short fibers form a certain shape. Therefore, the present method may enable us to determine the optimum curvilinear fiber path.

4 Conclusion

A design method is proposed for fibrous composite laminated plates with variable stiffness. The plate structure imitates the internal structures of bone or shell which possess variable stiffness in the natural world. In the present study, as an initial approach of designing the composite plates with continuous curvilinear fibers, the optimum short fiber distribution is determined.

The comparison between the variable stiffness plates with the optimally distributed short fibers and the conventional plates with the parallel fibers is made for maximum deflection. Then, the variable stiffness plates give lower maximum deflections for all examples. Therefore, it is concluded that the plate with optimally distributed short fibers may be designed to have higher stiffnesses than the conventional plates. Moreover, it is found that the optimum short fibers tend to allocate with a certain directional tendency. Hence, the plate with optimally continuous curvilinear fibers and having higher stiffness than the conventional plate may be fabricated using the result of the present study.

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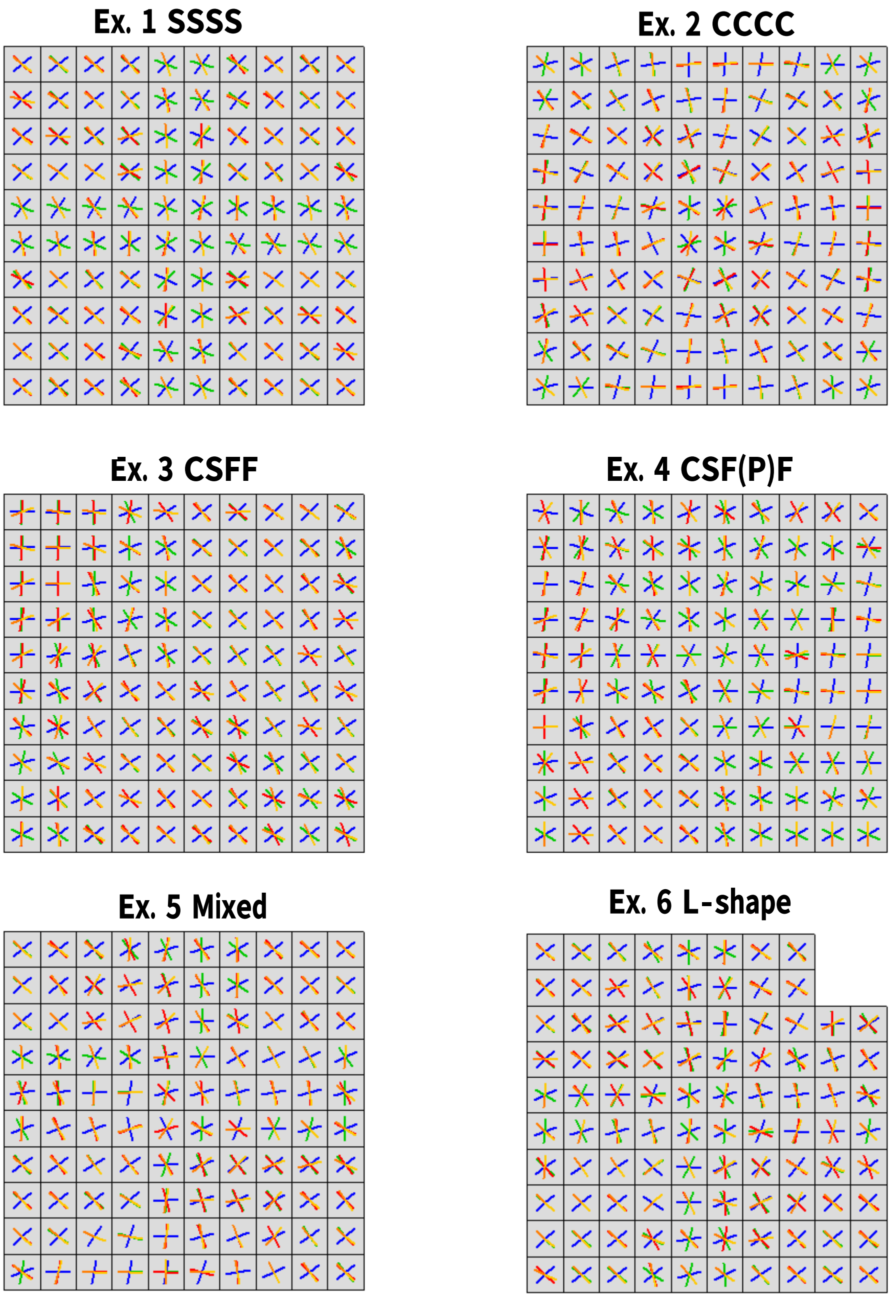

Fig. 6. Optimum short fiber distributions for symmetric 8-ply plates for six examples of boundary conditions

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