A THIN-WALLED COMPOSITE BEAM ELEMENT FOR OPEN AND CLOSED SECTIONS

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Abstract

An efficient beam element for the analysis of thin-walled laminated composite beams of open/closed sections is proposed. The cross-sectional stiffness of the beam is derived analytically where all possible couplings between torsion, bending and axial deformations are considered. In the finite element approximation of the beam, the torsional deformation requires a C^1 formulation due to incorporation of warping deformation while the bending deformation require a C^0 formulation due to the incorporation of shear deformation. The difficulty of implementation of both these formulations in the present coupled problem is successfully overcome by adopting an efficient approach for the finite element approximation of the bending deformations. The element is used for the numerical examples of open section I beam and closed section box beam problems having different boundary and loading conditions. The obtained results are compared with analytical/numerical and experimental results available in literature to demonstrate the performance of the proposed element.

1 Introduction

The modelling of thin-walled laminated composite beams and beam like slender structures, as an one dimensional condensed beam element, has drawn the attention of researchers for quite sometime, and a number of investigations have been carried out to study the different aspects of this. A few representative studies relevant to the present context are given in [1-8]. One of the initial applications of composite beam theory was found in the analysis of helicopter rotor blades. It has subsequently been applied to the analysis of pultruded composite profiles and other applications, including the analysis of long wind turbine blades made of composite materials.

The investigations carried out so far can be broadly divided into two groups, based on the approach followed to evaluate the constitutive matrix of the beam element, defined as the beam cross-section stiffness coefficients. The first and most common approach is based on an analytical technique, while the other approach requires a two-dimensional finite element analysis to obtain the cross-section stiffness matrix. Hodges and his co-workers [3] pioneered the second approach, which is defined as the so-called variational asymptotic beam section analysis (VABS). It is based on a method known as the variational asymptotic method (VAM) [9], where the three dimensional elasticity problem is systematically divided into a two dimensional cross-sectional problem, and a one dimensional beam problem. VABS has the advantage that beams having solid or thick-walled cross sections can be analyzed, where the three dimensional stresses can be extracted in the post processing stage. Opposed to this, the fully analytical approach may be preferred specifically for the analysis of beams with thin walled cross sections in order to avoid the additional two-dimensional finite element analysis required in VABS.

In the present study, a fully analytical (closed-form) approach is adopted for the derivation of the cross-sectional stiffness matrix considering different effects and their coupling to yield a very general formulation, which includes a torsional warping moment apart from the usual de St. Venant torsion contribution, axial force, bi-axial bending moments and transverse shear forces, thus yielding a 7x7 cross-sectional stiffness matrix. All the elements of this matrix are explicitly derived for open I section and closed box section profiles. For the constitutive equation of any beam wall, defined locally, provision is kept to enable the specification of either plane stress conditions (zero normal stress along the wall profile) or plane strain conditions (zero normal strain along the wall profile).

In the one dimensional finite element approximation, the torsional deformation requires C^1 continuity of the twisting rotation due to incorporation of the out of plane warping deformation. This requirement is satisfied with the use of a Hermetian interpolation function considering the twisting rotation and its derivative with respect to the length coordinate as the nodal unknowns. The association of the derivative of the twisting rotation helps to impose warping restraints or warping free conditions by constraining or releasing this nodal unknown. At the same time the bending deformation requires C^0 continuity of the transverse displacements due to the incorporation of the transverse shear deformation of the beam walls. A reduced integration technique is required for the evaluation of the stiffness matrix in order to avoid shear locking. As the bending deformation is not uncoupled from the other modes of deformation, including torsion, it is difficult to implement C^0 with a C^1 formulation having different integration schemes. Lee [7] tried to solve the problem by an amended representation of the torsional deformation, so as to model it with a C^0 formulation like bending deformation, but this involved a non-physical parameter in the formulation. Moreover, the C^0 formulation with reduced integration technique is susceptible to display inherent numerical disturbances like the occurrence of spurious modes.

Keeping these aspects in view, the finite element implementation of the bending deformation is carried out with a different approach based on the concept of the first author [10]. It does not require a reduced integration technique, which effectively eliminates the problem mention above. Based on this methodology a three node beam element, as shown in Fig. 1, has been developed, where the nodes at the two ends contain seven degrees of freedom (three translations, three rotations and the derivative of the twisting rotation), while the internal node contains five degrees of freedom (three translations and two bending rotations).

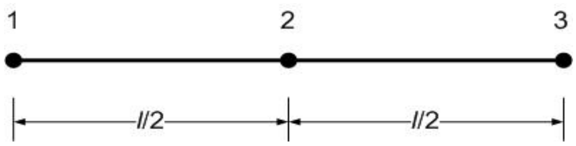

Fig. 1. Three node beam element

A computer code has been written in FORTRAN for the implementation of the element, which has been used to solve numerical examples of composite beams having open I and closed box sections. The results obtained in the form of deflections, angles of twist, and bending slopes are compared with analytical, experimental and/or other finite element analysis results available in literature. The results show a very good performance of the proposed element in terms of convergence and solution accuracy. The developed element is also utilized to derive some new results, which are presented for future references.

2 Formulation

A portion of the beam shell wall, with its local coordinate system *x*-*s*-*n* and displacement components, along with the global coordinate system *x*-*y*-*z* and displacement components, of the beam is shown in Fig. 2. In Fig. 2, *O* is the centroid, and *P* is the shear centre/pole of the beam section. The displacement components at mid-plane of the shell wall in the local coordinate system (*x*-*s*-*n*) may be expressed in terms of the global displacement components of the beam [1] as

$$\bar{u} = U + y\theta_y + z\theta_z + \varphi\theta_x' \tag{1}$$

$$\bar{v} = V\cos\alpha + W\sin\alpha - r\theta_x \tag{2}$$

$$\bar{w} = -V\sin\alpha + W\cos\alpha + q\theta_x \tag{3}$$

where φ is the warping function, $\theta_y = -V' + \Psi_y$ (V' is the derivative of V with respect to x and Ψ_y is the rotation of beam section about z for the transverse shear deformation) is the bending rotation of the beam section with respect to *z*, and $\theta_z = -W' + \Psi_z$ is the bending rotation of the beam section with respect to *y*. The corresponding displacement components at a point away from the shell mid-plane may be expressed as

$$u = \bar{u} + n\left(-\frac{\partial \bar{w}}{\partial x} + \psi_{xn}\right) \tag{4}$$

$$v = \bar{v} + n\left(-\frac{\partial \bar{w}}{\partial s} + \psi_{sn}\right) \tag{5}$$

$$w = \bar{w} \tag{6}$$

where ψ_{xn} and ψ_{sn} are the shear rotations of the shell sections about s and x, respectively, for transverse shear deformation. It is assumed that ψ_{sn} = 0 while ψ_{xn} may be expressed in terms of the corresponding global beam parameter as

$$\psi_{xn} = -\Psi_y \sin\alpha + \Psi_z \cos\alpha \tag{7}$$

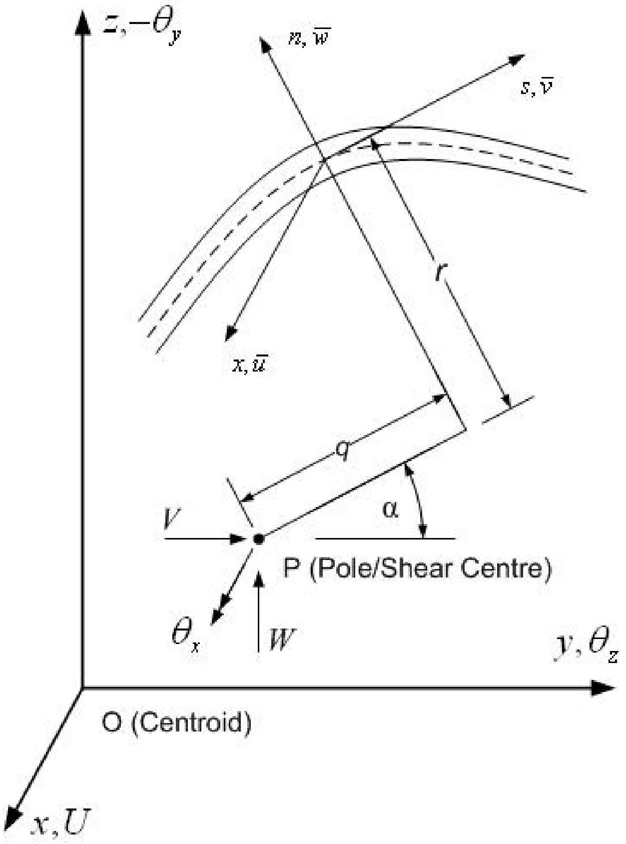

Fig. 2. Cross-section of a portion of shell wall of the thin-walled beam with coordinated systems

With the above equations, the displacement components at any point within the shell wall may be expressed in terms of the global beam displacement components as

$$\begin{aligned} u = U &+ (y - n\sin\alpha)\,\theta_y \\ &+ (z + n\cos\alpha)\,\theta_z + (\varphi - nq)\theta'_x \end{aligned} \tag{8}$$

$$v = V\cos\alpha + W\sin\alpha - (r + n)\,\theta_x \tag{9}$$

$$w = -V\sin\alpha + W\cos\alpha + q\theta_x \tag{10}$$

The strain components at the corresponding point in the local axis system (x-s-n) may be expressed as

$$\{\hat{\varepsilon}\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_s \\ \varepsilon_{xs} \\ \varepsilon_{xn} \\ \varepsilon_{sn} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial s} \\ \dfrac{\partial u}{\partial s} + \dfrac{\partial v}{\partial x} \\ \psi_{xn} \\ \psi_{sn} \end{Bmatrix} \tag{11}$$

Assuming the normal strain component ε_s to be zero (plane strain condition), or the normal stress component σ_s to be zero (plane stress condition), the reduced strain vector may be expressed in terms of the global displacement parameters with the help of the above equations (7-10) as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_{xs} \\ \varepsilon_{xn} \end{Bmatrix} = \begin{Bmatrix} \left(\dfrac{\partial u}{\partial x}\right) \\ \left(\dfrac{\partial u}{\partial s} + \dfrac{\partial v}{\partial x}\right) \\ \psi_{xn} \end{Bmatrix}$$

$$
= \left\{ \begin{array}{c} U' + (y - n\sin\alpha)\theta'_y + (z + n\cos\alpha)\theta'_z \\ +(\varphi - nq)\theta''_x \\ \Psi_y \cos\alpha + \Psi_z \sin\alpha - \left(2n + r - \dfrac{\partial\varphi}{\partial s}\right)\theta'_x \\ -\Psi_y \sin\alpha + \Psi_z \cos\alpha \end{array} \right\} \qquad (12)
$$

The above equation can further be rearranged in matrix form as

$$
\{\varepsilon\} = [H] \left\{ \begin{array}{c} U' \\ \theta'_y \\ \theta'_z \\ \theta''_x \\ \theta'_x \\ V' + \theta_y \\ W' + \theta_z \end{array} \right\} = [H]\{\bar{\varepsilon}\} \qquad (13)
$$

where

$$
[H]^T = \begin{bmatrix} 1 & 0 & 0 \\ y - n\sin\alpha & 0 & 0 \\ z + n\cos\alpha & 0 & 0 \\ \varphi - nq & 0 & 0 \\ 0 & -(2n + r - \varphi_{,s}) & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}
$$

The stress strain relationship considering the transverse shear deformation of the laminated shell wall [11] in the local coordinate system (x-s-n) may be expressed as

$$
\left\{ \begin{array}{c} \sigma_x \\ \sigma_s \\ \sigma_{xs} \\ \sigma_{xn} \\ \sigma_{sn} \end{array} \right\} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{61} & \bar{Q}_{62} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{55} & \bar{Q}_{54} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{44} \end{bmatrix} \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_s \\ \varepsilon_{xs} \\ \varepsilon_{xn} \\ \varepsilon_{sn} \end{array} \right\} \qquad (14)
$$

Assuming $\varepsilon_s = 0$ or $\sigma_s = 0$ along with $\psi_{ns} = 0$, the above equation reduces to

$$
\left\{ \begin{array}{c} \sigma_x \\ \sigma_{xs} \\ \sigma_{xn} \end{array} \right\} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{16} & 0 \\ \tilde{Q}_{61} & \tilde{Q}_{66} & 0 \\ 0 & 0 & \tilde{Q}_{55} \end{bmatrix} \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_{xs} \\ \varepsilon_{xn} \end{array} \right\} \qquad (15)
$$

where $\tilde{Q}_{11} = \bar{Q}_{11}$, $\tilde{Q}_{16} = \bar{Q}_{16}$, $\tilde{Q}_{66} = \bar{Q}_{66}$ and $\tilde{Q}_{55} = \bar{Q}_{55}$ for plane strain (ε_s = 0); and $\tilde{Q}_{11} = \bar{Q}_{11} - \bar{Q}_{12}\bar{Q}_{12}/\bar{Q}_{22}$, $\tilde{Q}_{16} = \bar{Q}_{16} - \bar{Q}_{12}\bar{Q}_{26}/\ \bar{Q}_{22}$, $\tilde{Q}_{66} = \bar{Q}_{66} - \bar{Q}_{16}\bar{Q}_{16}/\bar{Q}_{22}$ and $\tilde{Q}_{55} = \bar{Q}_{55}$ for plane stress (σ_s = 0).

Using equations (13) and (15) the strain energy of the system can be written as

$$
\tilde{U} = \frac{1}{2}\int \{\varepsilon\}^T \{\sigma\} dv = \frac{1}{2}\int \{\bar{\varepsilon}\}^T [D] \{\bar{\varepsilon}\} dx \qquad (16)
$$

where

$$
\begin{aligned} [D] &= \int\int [H]^T [\tilde{Q}] [H] dsdn \\ &= \int \left(\int [H]^T [\tilde{Q}] [H] dn \right) ds = \int [C] ds \end{aligned}
$$

All the elements of the cross-sectional stiffness matrix $[D]$ are explicitly derived for open I section and closed box section profiles. For this purpose the warping function φ used in the above equations are taken as

$$
\varphi = \int r ds - 2A_c \delta_s / \delta \qquad (17)
$$

where $\delta_s = \int \frac{ds}{\tilde{Q}_{66}}$, $\delta = \oint \frac{ds}{\tilde{Q}_{66}}$ and A_c is the cross-sectional area enclosed by the mid-plane contour (closed section). For an open section profile, the warping function may be simply obtained by dropping the second term associated with secondary warping, thus giving $\varphi = \int rds$.

For the one dimensional finite element implementation, a quadratic approximation has been adopted for the axial deformation, which follows a Lagrangian type formulation. The torsional deformation is based on a Hermetian formulation as mentioned earlier, where a cubic approximation has been adopted. The approximation of the bending deformation coupled with transverse shear deformation is based on the concept proposed by Sheikh [10]. Based on this the field variables are approximated as follows

$$U = a_1 + a_2 x + a_3 x^2 \tag{18}$$

$$V = a_4 + a_5 x + a_6 x^2 + a_7 x^3 \tag{19}$$

$$W = a_8 + a_9 x + a_{10} x^2 + a_{11} x^3 \tag{20}$$

$$\Psi_y = a_{12} + a_{13} x \tag{21}$$

$$\Psi_z = a_{14} + a_{15} x \tag{22}$$

$$\theta_x = a_{16} + a_{17} x + a_{18} x^2 + a_{19} x^3 \tag{23}$$

It should be noted that Ψ_y and Ψ_z are taken as the field variables instead of θ_y and θ_z, which are usually used in a typical C^0 formulation. It should be noted also that θ_y and θ_z appear as nodal unknowns instead of Ψ_y and Ψ_z. Now, with the help of equations (19-22), the bending rotations θ_y and θ_z may be expressed as

$$\theta_y = a_{12} + a_{13} x - a_5 - 2a_6 x - 3a_7 x^2 \quad (24$$

$$\theta_z = a_{14} + a_{15} x - a_9 - 2a_{10} x - 3a_{11} x^2 \tag{25}$$

The unknowns ($a_1, a_2, a_3, \cdots a_{19}$) in the above equations (18-23) are expressed in terms of the nodal displacement vector $\{\delta\}$ after substitution of U (eq. (18)), V (eq. (19)), W (eq. (20)), θ_y (eq. (24)) and θ_z (eq. (25)) at all three nodes of the beam element (see Fig. 1); and θ_x (eq. (23)) and its derivative θ'_x at the two external nodes as

$$\{\delta\} = [R]\{a\} \text{ or } \{a\} = [R]^{-1}\{\delta\} \tag{26}$$

where $\{a\} = [a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_{18} \quad a_{19}]^T$, $\{\delta\} = [U_1 \ V_1 \ W_1 \ \theta_{x1} \ \theta_{y1} \ \theta_{z1} \ \theta'_{x1} \ U_2 \ V_2 \ W_2 \ \theta_{y2} \ \theta_{z2} \ U_3 \ V_3 \ W_3 \ \theta_{x3} \ \theta_{y3} \ \theta_{z3} \ \theta'_{x3}]^T$, and the matrix $[R]$ of order of 19x19 contains the element nodal coordinates.

The generalized strain vector $\{\bar{\varepsilon}\}$ in equation (13) may be expressed in terms of $\{a\}$ using equations (18-23) as

$$\{\bar{\varepsilon}\} = \begin{Bmatrix} U' \\ \theta'_y \\ \theta'_z \\ \theta''_x \\ \theta'_x \\ V' + \theta_y \\ W' + \theta_z \end{Bmatrix} = \begin{Bmatrix} U' \\ -V'' + \Psi'_y \\ -W'' + \Psi'_z \\ \theta''_x \\ \theta'_x \\ \Psi'_y \\ \Psi'_z \end{Bmatrix} = [S]\{a\} \tag{27}$$

where the matrix $[S]$ of order of 7x19 is a function of x. The strain vector $\{\bar{\varepsilon}\}$ can be finally expressed in terms of the nodal displacement vector $\{\delta\}$ using equation (26) as

$$\{\bar{\varepsilon}\} = [S][R]^{-1}\{\delta\} = [B]\{\delta\} \tag{28}$$

With the above equation, the strain energy (eq. (16)) of the system may be expressed as

$$\tilde{U} = \frac{1}{2}\{\delta\}^T \int [B]^T [D][B] dx \{\delta\} = \frac{1}{2}\{\delta\}^T [K]\{\delta\} \quad (29)$$

where $[K]$ is the element stiffness matrix. Using equations (20) and (26), the element load vector due to a distributed transverse load of intensity q acting in the direction of z may be expressed as

$$\{P\} = [R]^{-T} \int q \left[S_q\right]^T dx \quad (30)$$

where the row matrix $\lfloor S_q \rfloor$ of 19 is function of x. The integrations involved in the evaluation of the element stiffness matrix $[K]$ and the load vector $\{P\}$ are carried out numerically following the Gauss quadrature technique.

3 Numerical Examples

In this section numerical examples of I and box beams are analysed using the proposed element, and the results obtained are compared with analytical, experimental and/or numerical results available in literature for most of the cases. The analysis is usually based on plane stress conditions, unless specified otherwise. In all examples the beam walls are assumed to be constituted by identical layers of identical thickness, but the layers may have different orientations. The geometry of the beam sections are defined in terms of centre line dimensions.

3.1 Simply supported I beam under uniformly distributed load

A 2.5m long open section I beam simply supported at its two ends, and subjected to a uniformly distributed transverse load of 1kN/m along the web of the beam, is analysed using the proposed element. The beam has a depth of 50mm, a flange width of 50mm and the same thickness of 2.08mm for the flanges and the web. The study is made with different symmetrical stacking sequences, where the flanges and the web are having identical lay-ups for all cases. The material properties of the layers are given in Table 1. The analysis is carried out assuming both plane stress and plane strain conditions. The values of deflection at the centre of the beam and the bending slope at its supports obtained in the present analysis are presented in Table 1 and Table 2, respectively. The results for the deflection are compared with those of Lee [7] and Lee and Lee [12] in Table 1. Lee [7] considered the effect of transverse shear deformation, whereas the study of Lee and Lee [12] is based on classical laminate theory. In both the studies [7, 12], one dimensional finite element analysis has been applied after obtaining the cross-sectional stiffness matrix analytically. For the validation of the results in [7,12], the beam structure was analysed using ABAQUS [13] where the S9R5 shell element was used to model the beam. The results produced by ABAQUS [13] are also included in Table 1. The table shows an excellent agreement of the present results with the other results, especially the results of Lee [7]. Moreover, the present results based on plane stress conditions are found to be closer to the results produced by ABAQUS [13]. It is also observed that only two elements could attain the convergence in all the cases.

Table 1. Deflection w (cm) at the centre of the simply supported I beam under uniform distributed loading (E_1 = 53.78GPa, E_2 = 17.93GPa, G_{12} = 8.96GPa, G_{13} = 8.96GPa, G_{23} = 3.45GPa, ν_{12} = 0.25)

Stacking sequence	$[0/0]_{4S}$	$[15/-15]_{4S}$	$[30/-30]_{4S}$
Present (σ_s = 0) - 2†	6.2641	6.9286	9.3195
Present (σ_s = 0) - 4	6.2641	6.9286	9.3195
Present (ε_s = 0) - 2	6.1342	6.6404	8.3092
Present (ε_s = 0) - 4	6.1342	6.6404	8.3092
Lee [7] (σ_s = 0)	6.259	6.923	9.314
Lee [7] (ε_s = 0)	6.129	6.637	8.307
Lee and Lee [12] (σ_s = 0)	6.233	6.899	9.290
Lee and Lee [12] (ε_s = 0)	6.103	6.610	8.281
ABAQUS [13]	6.340	6.989	9.360
Stacking sequence	$[45/-45]_{4S}$	$[60/-60]_{4S}$	$[75/-75]_{4S}$
Present (σ_s = 0) - 2	13.453	17.001	18.459
Present (σ_s = 0) - 4	13.453	17.001	18.459
Present (ε_s = 0) - 2	11.369	15.153	17.686
Present (ε_s = 0) - 4	11.369	15.153	17.686
Lee [7] (σ_s = 0)	13.446	16.992	18.449
Lee [7] (ε_s = 0)	11.363	15.151	17.683
Lee and Lee [12] (σ_s = 0)	13.421	16.962	18.411
Lee and Lee [12] (ε_s = 0)	11.340	15.119	17.643
ABAQUS [13]	13.479	17.023	18.490

† Number of elements

Table 2. Bending rotation θ_z x 100 (rad) at the support of the simply supported I beam under uniform distributed loading

Stacking sequence	$[0/0]_{4S}$	$[15/-15]_{4S}$	$[30/-30]_{4S}$
Present (σ_s = 0) - 2	7.9778	8.8301	11.892
Present (σ_s = 0) - 4	7.9778	8.8301	11.892
Present (ε_s = 0) - 2	7.8116	8.4613	10.600
Present (ε_s = 0) - 4	7.8116	8.4613	10.600
Stacking sequence	$[45/-45]_{4S}$	$[60/-60]_{4S}$	$[75/-75]_{4S}$
Present (σ_s = 0) - 2	17.179	21.711	23.567
Present (σ_s = 0) - 4	17.179	21.711	23.567
Present (ε_s = 0) - 2	14.515	19.353	22.582
Present (ε_s = 0) - 4	14.515	19.353	22.582

3.2 Clamped box beam under uniformly distributed load

A box beam clamped at both the ends, and subjected to uniformly distributed transverse load of 6.5kN/m along the mid-plane of one of the webs, is analysed using the proposed element assuming both plane stress and plane strain conditions. The beam is assumed to be free from axial and warping restraints at the supports. The beam is 1.0 m long, 70 mm deep and 50 mm wide, where all the beam walls are 2 mm thick having a stacking sequence of $(45/-45)_2/(0/0)_6/(45/-45)_2$. The material properties assumed for all layers are: E_1 = 148.0 GPa, E_2 = 9.65 GPa, G_{12} = G_{13} = G_{23} = 4.55 GPa, ν_{12} = 0.3. The values of the deflection and angle of twist at the centre of the beam obtained using the present analysis are presented in Table 3, along with results obtained by Kollar and Springer [5] assuming plane strain conditions, and Vo and Lee [14] assuming both plane stress and plane strain conditions. Kollar and Springer [5] solved the problem in closed form (analytically), while Vo and Lee [14] applied one dimensional finite element analysis after obtaining the cross-sectional stiffness matrix analytically. As both of the studies [5,14] did not consider the effect of transverse shear deformation, the present analysis was also carried out with a high value of transverse shear rigidity (G_{13} = G_{23} = G_{12} x 10^6), and the results obtained are included in Table 3 (marked ‡). The results show a significant effect of the transverse shear deformation. The table shows that the convergence of the present finite element formulation is good. It also shows a good agreement between the results obtained by the different techniques having similar basis.

Table 3. Deflection and angle of twist at the centre of the clamped box beam under uniform distributed loading

	w x 10^4 (m)	Θ_x x 10^3 (rad)
Present † (σ_s = 0) - 2	7.811	6.705
Present † (σ_s = 0) - 4	7.811	6.703
Present † (σ_s = 0) - 6	7.811	6.703
Present † (ε_s = 0) - 2	5.779	2.755
Present † (ε_s = 0) - 4	5.779	2.754
Present † (ε_s = 0) - 6	5.779	2.754
Present ‡ (σ_s = 0) - 2	4.940	6.705
Present ‡ (σ_s = 0) - 4	4.940	6.703
Present ‡ (σ_s = 0) - 6	4.940	6.703
Present ‡ (ε_s = 0) - 2	4.378	2.755
Present ‡ (ε_s = 0) - 4	4.378	2.754
Present ‡ (ε_s = 0) - 6	4.378	2.754
Vo and Lee [14] (σ_s = 0)	4.940	6.427
Vo and Lee [14] (ε_s = 0)	4.380	2.678
Kollar and Springer [5] (ε_s = 0)	4.880	2.760

† G_{13} = G_{12}, ‡ G_{13} = G_{23} = G_{12}, x 10^6

3.3 Cantilever I beam under tip load

A 30 inch long cantilever I beam subjected to a transverse unit load (1.0 lb) at the free end is analysed using the proposed element assuming restrained warping at both ends. The beam has a depth of 0.5 inch, a flange width of 1.0 inch. and same thickness (0.04 inch) is assumed for the flanges and the web. The stacking sequence of the flange is 0/90/0/90/90/0/15/15, while that of the web is 0/90/0/90/90/0/90/0. The assumed material properties are: E_1 = 20.59 x10^6 Psi, E_2 = 1.42 x10^6 Psi, G_{12} = G_{13} = G_{23} = 0.89 x10^6 Psi, ν_{12} = 0.42. The variation of deflection, bending slope and twisting rotation along the length of the beam are plotted in Fig. 3, Fig. 4 and Fig. 5, respectively. The results for the bending slope and twisting rotation are compared with the numerical results of Jung et al. [4] and the experimental results of Chandra and Chopra [15] in Fig. 4 and Fig. 5.

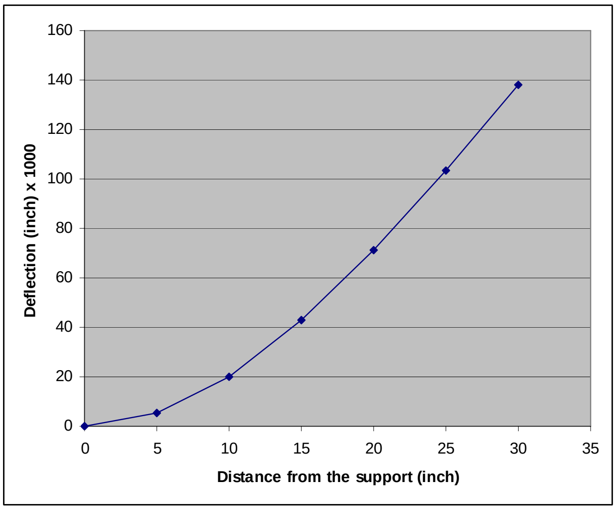

Fig. 3. Variation of deflection of the I beam

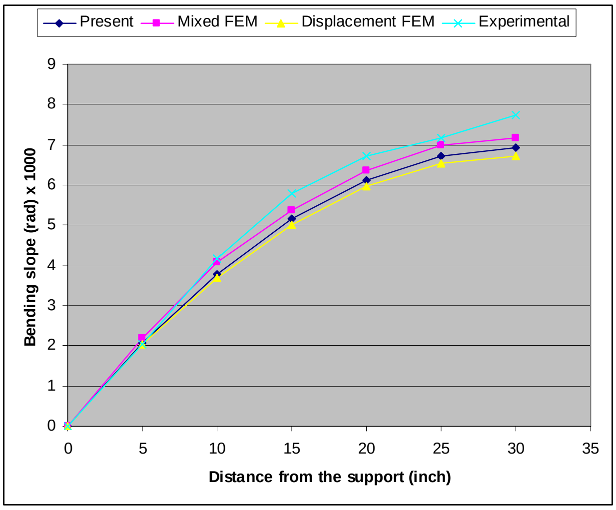

Fig. 4. Variation of bending slope of the I beam

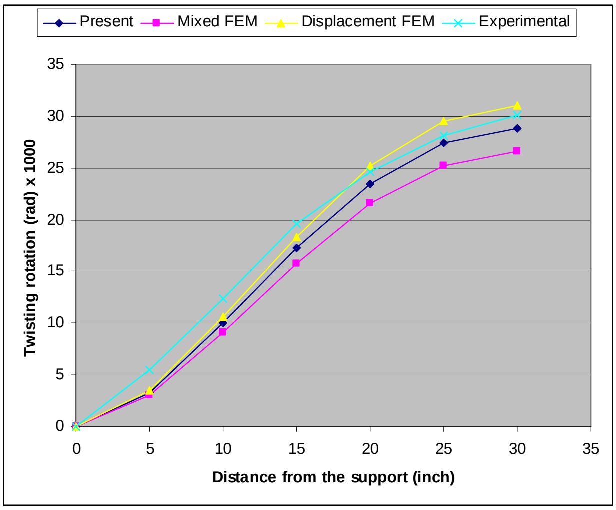

Fig. 5. Variation of twisting rotation of the I beam

Jung et al. [4] has produced results based on a mixed formulation, as well as on the displacement formulation of Smith and Chopra [16], where one dimensional finite element has been applied after getting the cross-sectional stiffness matrix analytically. The figures show a very good agreement between the results.

3.4 Cantilever box beam under tip load/twisting moment

A 30 inch long cantilever box beam having a depth of 0.5 inch, a width of 0.923 inch and assuming the same thickness (0.03 inch) for all the walls consisting of 6 layers is analysed using the proposed element.

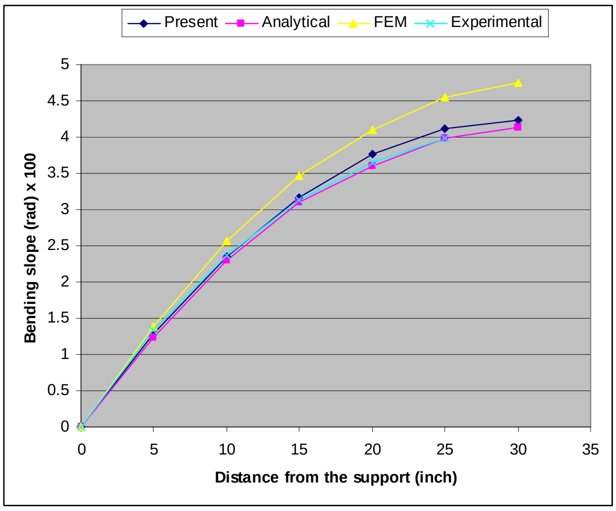

Fig. 6. Variation of bending slope of the box beam under a transverse load at the tip

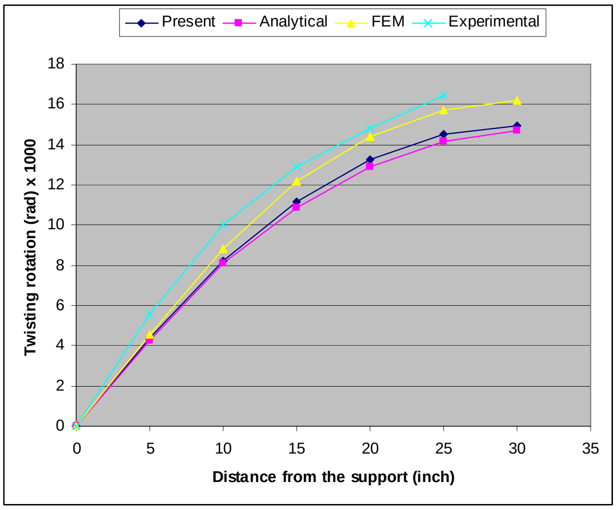

Fig. 7. Variation of twisting rotation of the box beam under a transverse load at the tip

The assumed loading is either a unit transverse load (1.0 lb) applied at the free end, or a unit twisting moment (1.0 lb-inch) applied at the free end. The stacking sequence of the top and bottom walls is $(45/45)_3$, while that of the left and right walls is $(45/-45)_3$. The material properties of all layers are assumed to be identical and equal to those adopted in the previous example. The variation of the bending slope and twisting rotation along the length of the beam are plotted in Fig. 6 to Fig. 9 along with the analytical and experimental results of Chandra et al. [2], and the finite element results of Stemple and Lee [17]. The results of the different methods are found to be in very good agreement.

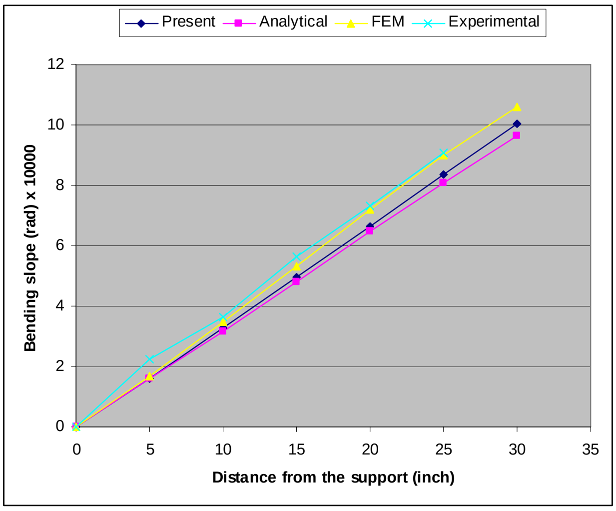

Fig. 8. Variation of bending slope of the box beam under a twisting moment at the tip

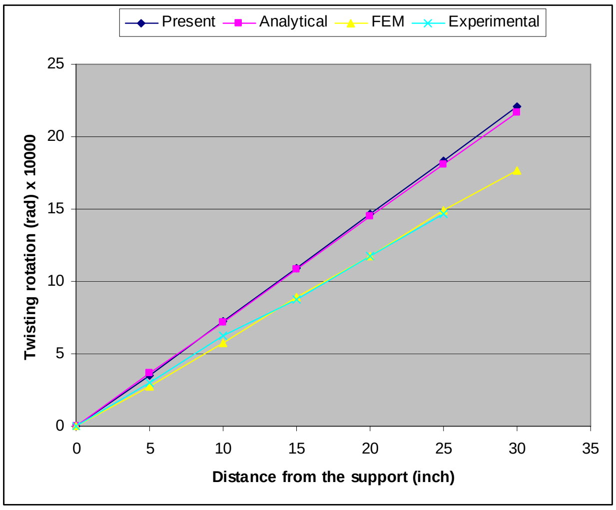

Fig. 9. Variation of twisting rotation of the box beam under a twisting moment at the tip

4 Conclusions

A fully coupled beam element has been developed for the analysis of thin-walled laminated composite beams of open and closed cross sections including axial displacement, torsion, out of plane warping, bi-axial bending and transverse shear deformation. The constitutive equations of the beam element are derived analytically considering the coupling of all these modes of deformation. The resulting composite beam theory is applied to open I section and closed box section beams. The incorporation of transverse shear deformation demands a C^0 formulation for the one dimensional finite element approximation of the bending deformations, while the torsional deformation demands a C^1 formulation for the incorporation of out of plane warping. The difficulty in implementing both formulations in the present coupled problem is successfully overcome by adopting an efficient approach for the finite element approximation of the bending deformations. Numerical examples of composite open and closed section beams having different load and boundary conditions are analysed using the proposed element. The results obtained are compared with analytical, experimental and/or other finite element results available in literature, and the comparisons show a very good performance of the proposed fully coupled beam element. Some new results are also presented for future references.

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