A NEW ANALYTICAL METHOD FOR PARTICULATE COMPOSITES

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Keywords: *Micromechanics, particulate composites, eigenfunctions, Green's function*

Abstract

A new approach to predict the mechanical behavior of general heterogeneous (composite) materials is presented. The eigenfunctions for the governing differential equation that the composite is subject to are derived in series format. The explicit form of the permissible functions that satisfy the continuity condition across the phase boundary is obtained with the help of a computer algebra system. The Green's function for the composite is constructed from the eigenfunctions. Using the Green's function, the physical field in the composite is expressed. Some numerical examples are shown.

Introduction

Finding physical fields such as stress and temperature distributions in general composite materials is a challenging task that has been studied for decades. The conventional macroscopic approach cannot account for intricate microstructures that are typical of modern composites where reinforcements include particulates, fibers and nanotubes. The pioneering work by Eshelby for an ellipsoidal inclusion inspired many researchers, which lead to a branch in applied mechanics called *micromechanics* [1]. The limitation of micromechanics is the widely used assumption that a composite must be infinitely extended and the distribution of reinforcements is statistically uniform. These assumptions severely limit the usefulness and finding physical fields such as stress and temperature distributions in general composite materials is a difficult task that has been studied for decades. The applicability of micromechanics to real-world composites is limited as the size of the actual composites is finite and the fiber distribution is not necessarily uniform. Purely numerical methods such as the finite element method or the boundary element method are routinely used for stress/thermal analysis of composites but their limitation as compared with micromechanics approaches is obvious. This paper addresses the shortcomings of both purely numerical and purely analytical approaches and introduces a semi-analytical approach to general heterogeneous materials. Permissible functions that satisfy the continuity conditions across the phase interface of a composite are shown to be derived using a computer algebra system and they are used to construct the eigenfunctions for the corresponding Sturm-Liouville system. Examples of eigenfunctions for a medium that contains a spherical inclusion will be presented.

Formulations

The following formulations are written in terms of the quantities in solid mechanics but they can be modified to those equations in steady state heat conduction in composites as well. The static stress-equilibrium equation for the displacement field, u, is expressed as

$$\nabla \cdot \left(C(x) \nabla u(x) \right) + b(x) = 0, \quad (1)$$

where $C(x)$ is the elastic modulus (a function of position), $\nabla \cdot$ is the divergence operator, ∇ is the gradient operator and b is a body force. The symbol, x, denotes the position vector. Equation (1) along with a prescribed boundary condition constitutes a boundary value problem. The solution to equation (1) can be expressed if the Green's function, $g(x, x')$, defined as

$$\nabla \cdot \left(C(x) \nabla g(x, x') \right) + \delta(x - x') = 0, \quad (2)$$

is known where $\delta(x - x')$ is the Dirac delta function [2]. Using equation (2), the displacement field is expressed as

$$u(x) = u^o(x) + \int g(x, x') b(x') dx', \quad (3)$$

where u^0 is a term that satisfies the prescribed boundary displacement condition. The stress field is then expressed as

$$\sigma(x) = \sigma^o(x) + C(x)\int \nabla g(x,x')b(x')dx', \qquad (4)$$

where $\sigma^o(x)$ is the stress field at the boundary. The solution to equation (2) is available for only a few simple cases. For example, if the medium is homogeneous, isotropic and extended to infinity, the Green's function is expressed as

$$g_{im}(\mathbf{x}) = \frac{1}{8\pi\mu(2\mu+\lambda)}\left\{(3\mu+\lambda)\frac{\delta_{ij}}{r} + (\mu+\lambda)\frac{x_i x_j}{r^3}\right\}, \qquad (6)$$

where μ and λ are the Lame constants and $r = |x|$. There is no exact solution available when the medium is inhomogeneous or finite.

It is possible to express the Green's function in a series form as [3]

$$g_{km}(x,x') = \sum_{\alpha}^{\infty} \frac{\phi_m^\alpha(x)\phi_k^\alpha(x')}{\lambda^\alpha}, \qquad (7)$$

where $\phi_m^\alpha(x)$ and λ^α are the eigenfunction and the eigenvalue defined as

$$(C_{ijkl}(x)\phi_{k,l}^\alpha(x))_{,j} + \lambda^\alpha\phi_i^\alpha(x) = 0. \qquad (8)$$

In equation (8), the repeated indices denote summation. The eigenfunctions are mutually orthogonal as

$$\int_D \phi_m^\alpha(x)\phi_n^\beta(x)dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \delta_{mn} & \text{if } \alpha = \beta \end{cases} \qquad (9)$$

where δ_{mn} is the Kronecker delta.

Solving equation (8) for $\phi_i^\alpha(r)$ is just as difficult as solving equation (2). Therefore, an approximate solution to equation (8) is sought. In this approximation, the eigenfunction, $\phi_k^\alpha(x)$, is to be approximated as a series of permissible functions as

$$\phi_k^\alpha(x) = \sum_\beta c_k^{\alpha\beta} f^\beta(x), \qquad (10)$$

where $f^\beta(x)$ is a permissible function chosen from elementary functions to satisfy the continuity condition of the displacement and traction force across the interface as well as the homogeneous boundary condition. The quantity, $c_k^{\alpha\beta}$, is the k-th coordinate component of the coefficient of $f^\beta(x)$ of the α-th eigenfunction. The Galerkin method is used here to determine $c_k^{\alpha\beta}$ [4]. By substituting equation (10) into equation (8), multiplying $f^\alpha(x)$ on the both sides and integrate them over the entire domain, equation (8) can be converted to the following algebraic eigenvalue equation:

$$\sum_\beta A_{ik}^{\gamma\beta} c_k^{\alpha\beta} + \lambda^\alpha \sum_\beta B^{\gamma\beta} c_i^{\alpha\beta} = 0, \qquad (11)$$

where

$$A_{ik}^{\gamma\beta} = \int_D (C_{ijkl} f^\beta(x)_{,l})_{,j} f^\gamma(x)dx, \qquad (12)$$

and

$$B^{\gamma\beta} = \int_D f^\beta(x) f^\gamma(x)dx. \qquad (13)$$

Equation (11) is an algebraic eigenvalue problem that can be solved on a routine basis once the components of the matrices, $A_{ik}^{\gamma\beta}$ and $B^{\gamma\beta}$, are computed. Obviously, such a task requires an enormous amount of algebra and is best carried out by a computer algebra system.

Example

A proper choice of permissible functions is essential for faster convergence of the solution. Permissible functions are chosen so that they satisfy the given homogeneous boundary condition as well as the continuity condition across the phase boundary. For example, if a 2-D body is homogeneous and its boundary is rectangular, a permissible function can be chosen as

$$f(x,y) = \sum_{i,j} a_{ij}(x-a)^2(y-b)^2 x^i y^j, \qquad (14)$$

If a 2-D medium is extended to infinity, a permissible function can be chosen as

$$f(x,y) = \sum_{i,j} a_{ij} e^{-x^2-y^2} x^i y^j, \qquad (15)$$

Note that equation (15) vanishes at infinity.

If a 2-D medium has an inclusion with a different material property, a permissible function can be set up for each phase. For instance, one of the permissible functions for an infinitely extended medium (k_2) with a circular inclusion (k_1) having a radius of a is expressed as

$$f(x,y)=\begin{cases} -a^2x-\dfrac{2a^2k_1x}{k_1-k_2+a^2k_2}+ & x^2+y^2<a^2 \\ x^3+xy^2 & \\ -\dfrac{2a^2e^{a^2/2-1/2(x^2+y^2)}k_1x}{k_1-k_2+a^2k_2} & x^2+y^2>a^2 \end{cases} \tag{16}$$

Note that equation (16) satisfies the following continuity conditions:

$$f_1=f_2, \tag{17}$$

$$k_1\frac{\partial f_1}{\partial n}=k_2\frac{\partial f_2}{\partial n}, \tag{18}$$

$$f_2=f_m, \tag{19}$$

$$k_2\frac{\partial f_2}{\partial n}=k_m\frac{\partial f_m}{\partial n}. \tag{20}$$

Using permissible functions, the eigenfunctions, $\varphi(x)$, can be obtained by their combinations. The following two graphs are the first two eigenfunctions for the same medium used by equation (16). It should be noted that the two eigenfunctions are mutually orthogonal each other.

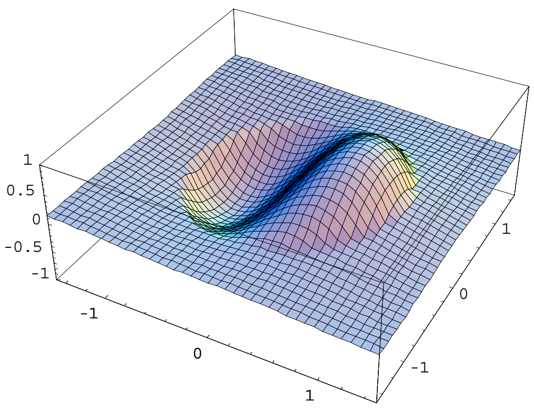

Fig. 1. Eigenfunction 1

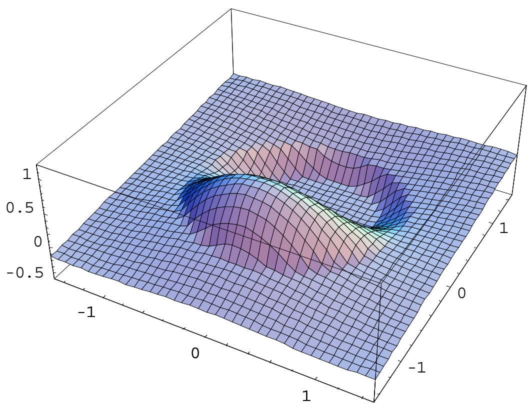

Fig. 2. Eigenfunction 2

Using the eigenfunctions, the Green's function can be then expressed by equation (7). With the Green's function, a convolution type of integrals can express the physical field. Results will be reported elsewhere.

Conclusions

An analytical procedure was introduced to proper choice of permissible functions is important for faster convergence. Permissible functions are chosen so that they satisfy the given homogeneous boundary condition as well as the continuity condition. Application to other types of heterogeneous materials will be investigated.

References

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