MIXED MODE I AND II STRESS INTENSITY FACTORS FOR EDGE CRACKS IN ANISOTROPIC INFINITE STRIPS

Haiying Huang and George A. Kardomateas

School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332-0150 USA

SUMMARY: The Mode I and II stress intensity factors in a fully anisotropic infinite strip with a single-edge or double-edge crack configuration are obtained from an approach based on the continuous dislocation technique. The elastic solution of a single dislocation in an anisotropic half plane is used in conjunction with an array of dislocations along the boundary of the infinite strip, which is supposed to be traction-free, to provide the solution of a single dislocation in an anisotropic infinite strip. This solution is then applied to calculate the mixed mode I and II stress intensity factors for a single-edge and a double-edge crack in the anisotropic strip, by replacing the cracks with a series of dislocations and satisfying the crack surface traction-free conditions. Results are presented for a graphite/epoxy in a unidirectional construction with the fiber orientation varying between 0 and 90 degrees.

INTRODUCTION

Very few papers have dealt with non-isotropic cracks in finite or semi-infinite bodies. An edge crack in a strip with edge loading was studied by Thouless et al [1] and by Suo [2], whereas as far as solutions in orthotropic materials, the distributed dislocation method was employed by Suo [2] and Suo and Hutchinson [3] to obtain the stress intensity factors for an orthotropic strip. The method employed for the field in the unflawed strip was an Airy stress function solution with Fourier transforms. Suo [2] also discussed the possible extension to an anisotropic strip. In earlier studies [4], an orthotropic infinite strip with a semi-infinite crack mid-distance of the strip faces was investigated by using Fourier transforms in combination with the Wiener-Hopf technique. In a more recent study,Qian and Sun [5] obtained stress intensity factors for interface cracks between two monoclinic media, by either calculating the finite-extension strain energy release rates or utilizing the relationships between the crack surface displacements and the stress intensity factors, both carried out with a finite element analysis.

A solution to the problem of a fully anisotropic strip with a single-edge or double-edge cracks is presented in this paper. A different approach than the Fourier transform method of Suo [2] or the finite element approach of Qian and Sun [5] is followed in this paper. First, the elastic solution of a dislocation in an anisotropic infinite strip is derived by assigning an array of dislocations along a line in the half plane which is supposed to be the boundary of the infinite strip. The dislocation densities of these added dislocations are determined by satisfying the traction-free boundary conditions. The stress fields of a single dislocation in the anisotropic strip is thus the combination of that of the single dislocation and those of the dislocations distributed along the boundary. Subsequently, the elastic solution is employed to calculate the mixed-mode stress intensity factors of single- and double- edge cracks in the anisotropic infinite strip. The material anisotropy and the crack length effects on the Mode I and II stress intensity factors are investigated.

Formulation

A Dislocation in a Fully Anisotropic Strip

The geometry of a dislocation in an infinite strip can be decomposed into two geometries. The first one is a half plane with a single dislocation located at point (x_0, y_0) for which the elastic solution of the dislocation can be found in Lee [6]. In short, the stress components at point (x,y) due to a dislocation $B=B_x+iB_y$ located at (x_0, y_0) can be expressed as:

$$\sigma_{ij}(x,y)=B_x(x_0,y_0)G_{xij}(x,y,x_0,y_0)+B_y(x_0,y_0)G_{yij}(x,y,x_0,y_0), \tag{1}$$

where $ij=xx$, yy, xy and $G_{xij}(x, y, x_0, y_0)$ are the stress components at (x, y) due to a unit dislocation $B_x=1$ at (x_0, y_0) and $G_{yij}(x, y, x_0, y_0)$ are the stress components at (x, y) due to a unit dislocation $B_y=1$ at (x_0, y_0).

Accordingly, the tractions along the dashed line, which is supposed to be the boundary of the infinite strip, due to the single dislocation $B(x_0, y_0)$ in the half plane are found from (1) by setting $y=h$.

The second geometry is a half plane with an array of dislocations along the line which is supposed to be the boundary of the strip. The dislocation densities of the dislocations $b(x, h)$ along the boundary are determined in such a way that the tractions generated by these dislocations along the boundary line $\sigma_{yy}^{(a)}(x,h)$ and $\sigma_{xy}^{(a)}(x,h)$ are the opposite of $\sigma_{yy}^{(s)}(x,h)$ and $\sigma_{xy}^{(s)}(x,h)$. Thus the traction-free boundary conditions of the infinite strip are satisfied after superposing these two geometries together. Suppose that the dislocation array $b(t, h)=b_x(t,h)+ib_y(t,h)$ is distributed from $-\infty$ to ∞, then the tractions along the boundary are in the form:

$$\sigma_{yy}^{(a)}(x,h)=\int_{-\infty}^{\infty}\left[b_x(t,h)G_{xyy}(x,h,t,h)+b_y(t,h)G_{yyy}(x,h,t,h)\right]dt=-\sigma_{yy}^{(s)}(x,h). \tag{2}$$

The functions $G_{xyy}(x, h, t, h)$, $G_{yyy}(x, h, t, h)$, $G_{xxy}(x, h, t, h)$ and $G_{yxy}(x, h, t, h)$ are singular at $x=t$. Since the single dislocation is in self- equilibrium, the tractions $\sigma_{yy}^{(s)}(x,h)$ and $\sigma_{xy}^{(s)}(x,h)$ vanish as $t\to-\infty,+\infty$. As a result, the dislocation densities $b_x(t,h)$ and $b_y(t,h)$ go to zero as $t\to-\infty,+\infty$. Therefore, for calculation purposes, we can ignore the dislocations located at $x>d$ and $x<-d$, where d is a value large enough compared to h. A value of $d=100h$ was found to be more than adequate for this purpose.

If we use the normalization:

$$\tilde{t}=\frac{t}{d}\ \text{and}\ \tilde{x}=\frac{x}{d}$$

and enforce that $b(\tilde{t},h)=b_x(\tilde{t},h)+ib_y(\tilde{t},h)$ be zero at $\tilde{t}=-1$ and $\tilde{t}=1$, which can be built into the solution by expressing $b(\tilde{t},h)$ as the product of a fundamental function $W(\tilde{t})$ and an unknown function $\tilde{b}(\tilde{t},h)$ [7]:

$$b(\tilde{t}) = W(\tilde{t})\tilde{b}(\tilde{t},h); \quad W(\tilde{t}) = \sqrt{1-\tilde{t}^2}\ ,$$

we then obtain the numerical form of the singular integral equations:

$$\pi d\left[\sum_{i=1}^{N} W_i\tilde{b}_x(\tilde{t}_i,h)G_{xij}(x_k,h,t_i,h) + \sum_{i=1}^{N} W_i\tilde{b}_y(\tilde{t}_i,h)G_{yij}(x_k,h,t_i,h)\right] = \\ = -\sigma_{ij}^{(s)}(x_k,h), \qquad ij = xx, xy \quad k = 1,\ldots,N+1 \tag{3}$$

where $\tilde{t}_i$ are the N discrete integral points and $\tilde{x}_k$ are the collocation points and W_i are the weight coefficients:

$$\tilde{t}_i = \cos\left(\frac{\pi i}{N+1}\right); \quad \tilde{x}_k = \cos\left[\frac{\pi(2k-1)}{2(N+1)}\right]; \quad W_i = \frac{1-\tilde{t}_i^2}{N+1}$$

Equation (3) allows us to determine the dislocation densities $b(\tilde{t}_i,h)$ of the dislocation array along the dashed line, which cancel out the residual tractions due to a single dislocation $B(x_0\ ,y_0)$ in the first geometry.

Therefore, the stress components at (x, y) due to a dislocation $B\,(x_0\ ,y_0)=B_x(x_0\ ,y_0)+iB_y(x_0, y_0)$ in the infinite strip are in the form:

$$\sigma_{ij}(x,y) = B_x(x_0,y_0)\tilde{G}_{xij}(x,y,x_0,y_0) + B_y(x_0,y_0)\tilde{G}_{yij}(x,y,x_0,y_0). \tag{4}$$

Physically, $\tilde{G}_{xij}(x,y)$ represent the stresses at (x,y) due to a dislocation $B_x=1$ at (x_0, y_0) in the infinite strip. Similarly, $\tilde{G}_{yij}(x,y)$ represent the stresses at (x,y) due to a dislocation $B_y=1$ at (x_0, y_0) in the infinite strip. For more detailed expressions of these functions, refer to Huang and Kardomateas [8].

A Single Edge Crack Under Uniform Tension

An infinite strip with an edge crack subjected to uniform tension is considered next; the edge crack, of length a, is located in the lower half of the infinite strip and is aligned with the y axis. Thus, the crack tip is at $y=a$. Replacing the crack with a series of dislocations, we find that the singular integral equations that ensure crack surface traction-free condition for the edge crack are in the form:

$$\sigma_{ij}^{(d)}(O,y) = \int_0^a \left[B_x(0,t)\tilde{G}_{xij}(0,y,0,t) + B_y(0,t)\tilde{G}_{yij}(0,y,0,t)\right]dt = -\sigma_0, \tag{5}$$

The foregoing equations ensure that the tractions $\sigma_{xx}^{(d)}$ and $\sigma_{xy}^{(d)}$ cancel out the tractions along the crack face due to the external loading, which is σ_0 in our case. It should be mentioned that the edge crack is a surface-breaking crack and the stress components are not singular at both ends of the crack. The Gaussian quadrature, which would be employed to solve the singular integral equations, has to be chosen carefully so that it includes all the appropriate end-point asymptotics. First the integral equations are normalized through the following substitutions:

$$\bar{t} = \frac{2t-a}{a}, \quad \bar{y} = \frac{2y-a}{a}$$

Now, $B_x(\bar{t})$ and $B_y(\bar{t})$ must be singular at the crack tips $\bar{t}=1$ $(t=a)$ and bounded at $\bar{t}=-1$ $(t=0)$. Thus, the dislocation density $B(\bar{t})$ should be expressed in the following product of a fundamental function $W(\bar{t})$ and an unknown regular function $\widetilde{B}(\bar{t})$:

$$B(\bar{t}) = W(\bar{t})\widetilde{B}(\bar{t}); \quad W(\bar{t}) = \sqrt{\frac{1+\bar{t}}{1-\bar{t}}}$$

The resulting numerical form of the singular integral equations is in the form:

$$\pi a\left\{\sum_{i=1}^{N} W_i \widetilde{B}_x(0,\bar{t}_i)\widetilde{G}_{xxx}(0,y_k,0,t_i) + \sum_{i=1}^{N} W_i \widetilde{B}_y(0,\bar{t}_i)\widetilde{G}_{yxx}(0,y_k,0,t_i)\right\} = \tag{6}$$

$$= -\sigma_0 \qquad k = 1 \ldots N$$

where $\bar{t}_i$ are the N discrete integration points and $\bar{y}_k$ are the N collocation points and W_i are weight coefficients:

$$\bar{t}_i = \cos\left(\pi\frac{2i-1}{2N+1}\right); \quad \bar{y}_k = \cos\left(\pi\frac{2k}{2N+1}\right); \quad W_i = \frac{2(1+\bar{t}_i)}{2N+1}.$$

As there are 2N collocation points and 2N integral points, these eqs. are sufficient for the determination of the dislocation density $\widetilde{B}(0,\bar{t}_i) = \widetilde{B}_x(0,\bar{t}_i) + i\widetilde{B}_y(0,\bar{t}_i)$ along the edge crack. Of major significance is the value of the dislocation density at the crack tip, $\widetilde{B}(+1)$, as it is directly related to the stress intensity factors. It can be obtained from Krenk's interpolation formulae [7].

The stress intensity factor at the crack tip $y=a$, is defined as:

$$K_I + iK_{II} = \lim_{y\to a}\left\{\sqrt{2\pi(a-y)}\left[\sigma_{xx}(y) + i\tau_{xy}(y)\right]_{x=0}\right\}.$$

Using relations for the stresses in terms of the complex potentials, gives

$$K_I + iK_{II} = \lim_{y\to a}\left\{\sqrt{2\pi(a-y)}\sum_{j=1,2}\left(\mu_j^2 - i\mu_j\right)\phi_j'(z_j) + \left(\bar{\mu}_j^2 - i\bar{\mu}_j\right)\bar{\phi}_j'(z_j)\right\}.$$

Only the singular part of the stress potential contributes, which is due to a dislocation at the crack tip, i.e. as $y \to a$ and $z \to z_0$, and, finally, after algebraic manipulation (for details refer to Huang and Kardomateas, [8]) the stress intensity factors at the crack tip $y = a$ are related to the dislocation densities at the crack tip as follows:

$$K_I + iK_{II} = \sqrt{2\pi a}\,\{\left[(\mu_1 - i)A_{11} + (\mu_2 - i)A_{21} + (\bar{\mu}_1 - i)\bar{A}_{12} + (\bar{\mu}_2 - i)\bar{A}_{22}\right]\widetilde{B}(+1) +$$

$$\left[(\mu_1 - i)A_{12} + (\mu_2 - i)A_{22} + (\bar{\mu}_1 - i)\bar{A}_{11} + (\bar{\mu}_2 - i)\bar{A}_{21}\right]\overline{\widetilde{B}(+1)}\}, \tag{7}$$

where A_{11}, A_{12}, A_{21}, A_{22} are defined in terms of the anisotropic material properties. Notice that $\overline{\widetilde{B}(+1)}$ is the complex conjugate of $B(+1)$.

Double Edge Cracks Under Uniform Tension

Another geometry we studied is a rectangular plate with double edge cracks subjected to uniform external load. Two edge cracks are of length a and located symmetrically about the middle plane of the infinite strip. Both cracks are aligned with the y axis. The lower edge crack is denoted as crack I and the upper edge crack as crack II.

The crack surface traction-free conditions in eq. (5) is for a single edge crack and can be easily extended to the case of double edge cracks. Choosing coordinate systems for edge crack I and II as $y^{(i)}$ and $t^{(i)}$ (i=1,2), the expressions for the crack surface tractions yield systems of singular integral equations, which include the dislocation densities from both cracks.

The following normalization is used:

$$\bar{y}^{(1)} = \frac{2y^{(1)}}{a} - 1, \quad \bar{t}^{(1)} = \frac{2t^{(1)}}{a} - 1,$$

and

$$\bar{y}^{(2)} = \frac{2y^{(2)}}{a} - \left(\frac{2h}{a} - 1\right), \quad \bar{t}^{(2)} = \frac{2t^{(2)}}{a} - \left(\frac{2h}{a} - 1\right).$$

Now, $B_x\left(\bar{t}^{(1)}\right)$ and $B_y\left(\bar{t}^{(1)}\right)$ must be singular at the crack tip $\bar{t}^{(1)} = 1\left(t^{(1)} = a\right)$ and bounded at the edge $\bar{t}^{(1)} = -1\left(t^{(1)} = 0\right)$. The form of $B_x\left(\bar{t}^{(2)}\right)$ and $b_y\left(\bar{t}^{(2)}\right)$ are the opposite of $B_x(\bar{t}^{(1)})$ and $b_y(\bar{t}^{(1)})$, i.e., $B_x\left(\bar{t}^{(2)}\right)$ and $B_y\left(\bar{t}^{(2)}\right)$ are bounded at $\bar{t}^{(2)} = 1(t^{(2)} = h)$ and singular at $\bar{t}^{(2)} = -1\left(t^{(2)} = h - a\right)$. Expressing $B\left(\bar{t}^{(1)}\right)$ and $B(\bar{t}^{(2)})$ as

$$B\left(\bar{t}^{(1)}\right) = W^{(1)}\left(\bar{t}^{(1)}\right)\widetilde{B}\left(\bar{t}^{(1)}\right); \quad W^{(1)}\left(\bar{t}^{(1)}\right) = \sqrt{\frac{1+\bar{t}^{(1)}}{1-\bar{t}^{(1)}}},$$

$$B\left(\bar{t}^{(2)}\right) = W^{(2)}\left(\bar{t}^{(2)}\right)\widetilde{B}\left(\bar{t}^{(2)}\right); \quad W^{(2)}\left(\bar{t}^{(2)}\right) = \sqrt{\frac{1+\bar{t}^{(2)}}{1-\bar{t}^{(2)}}},$$

where $\bar{t}_i^{(j)}$ are the $2N$ discrete integration points; $\bar{y}_k^{(m)}$ are the $2N$ collocation points and $W_i^{(j)}$ are weight coefficients, gives the numerical form of the singular integral equations as:

$$\pi a \sum_{j=1}^{2}\left\{\sum_{i=1}^{N} W_i^{(j)}\widetilde{B}_x\left(\bar{t}_i^{(j)}\right)\widetilde{G}_{xij}\left(y_k^{(m)}, t_i^{(j)}\right) + \sum_{i=1}^{N} W_i^{(j)}\widetilde{B}_y\left(\bar{t}_i^{(j)}\right)\widetilde{G}_{yij}\left(y_k^{(m)}, t_i^{(j)}\right)\right\} =$$
$$= -\sigma_{ij}^{(\sigma)}\left(y_k^{(m)}\right) \qquad k = 1 \ldots N, \quad m = 1,2 \text{ and } ij = xx, xy \tag{8}$$

Now we have $4N$ linear equations to solve for the $4N$ unknowns, i.e., $\widetilde{B}^{(1)}\left(\bar{t}_i^{(1)}\right) = \widetilde{B}_x\left(\bar{t}_i^{(1)}\right) + i\widetilde{B}_y\left(\bar{t}_i^{(1)}\right)$ and $\widetilde{B}^{(2)}(\bar{t}_i^{(2)}) = \widetilde{B}_x(\bar{t}_i^{(2)}) + i\widetilde{B}_y(\bar{t}_i^{(2)})$ can be solved at the discrete set of points $\bar{t}_i^{(1)}$ and $\bar{t}_i^{(2)}$ from eqs (8). Again, the value of $\widetilde{B}^{(1)}(+1)$ and $\widetilde{B}^{(2)}(-1)$ can be obtained from Krenk's interpolation formulae [7].

In a similar fashion to the single edge crack, the stress intensity factors at $y = a$ can be related to the dislocation densities from the following expression:

$$(K_I + iK_{II})\big|_a =$$

$$\sqrt{2\pi a}\,\{[(\mu_1 - i)A_{11} + (\mu_2 - i)A_{21} + (\overline{\mu}_1 - i)\overline{A}_{12} + (\overline{\mu}_2 - i)\overline{A}_{22}]B_z^{(1)}(+1)$$
$$+[(\mu_1 - i)A_{12} + (\mu_2 - i)A_{22} + (\overline{\mu}_1 - i)\overline{A}_{11} + (\overline{\mu}_2 - i)\overline{A}_{21}]\overline{B}_z^{(1)}(+1)\}, \quad (9a)$$

Similarly, the stress intensity factors at the other crack tip, $y = h\text{-}a$, are:

$$(K_I + iK_{II})\big|_{h-a} =$$

$$-\sqrt{2\pi a}\,\{[(\mu_1 - i)A_{11} + (\mu_2 - i)A_{21} + (\overline{\mu}_1 - i)\overline{A}_{12} + (\overline{\mu}_2 - i)\overline{A}_{22}]B_z^{(2)}(-1)$$
$$+[(\mu_1 - i)A_{12} + (\mu_2 - i)A_{22} + (\overline{\mu}_1 - i)\overline{A}_{11} + (\overline{\mu}_2 - i)\overline{A}_{21}]\overline{B}_z^{(2)}(-1)\}. \quad (9b)$$

Discussion of Results

The effects of material anisotropy on the mode I stress intensity factors and the mode mixity for a single edge crack under uniform tension are shown in Figures 1a and 1b. Typical data for graphite/epoxy were used, i.e., moduli in GPa: E_L=130, E_T = 10.5, G_{LT} = 6 and Poisson's ratio $\upsilon_{LT} = 0.28$, where L and T are the directions along and perpendicular to the fibers, respectively. A unidirectional construction was considered with the fiber orientation angle, θ, varying from 0 to 90 degrees. The orientation angle θ is measured from the x direction, i.e. θ = 0° is when the crack is perpendicular to the fibers and θ = 90° is when the crack is parallel with the fibers. Obviously, the limits of θ = 0° and 90° are the orthotropic cases. Both the normalized mode I stress intensity factors and the mode mixities increase as the relative length $\alpha = a/h$ increases. Because the crack is not symmetric, the mode I stress intensity factors at fiber orientation 0° and 90° are different and the difference is more obvious for longer cracks. In addition, the anisotropic single edge crack is under mixed-mode loading even though the external load is uniform and the crack is relatively short. As shown in Figure 1a, the effect of anisotropy on the mode I stress intensity factors is seen to be significant between 30 and 60 degrees and depends also on the relative crack length $a = a/h$, being larger for cracks of relative larger length. The mode mixity ψ in Figure 1b is defined as

$$\psi = \tan^{-1}\left(\frac{K_{II}}{K_I}\right),$$

and expresses the relative amounts of mode I and mode II components. As expected, the mode II stress intensity factors are zero for orthotropic materials. The effect of anisotropy on the mode mixity is dependent on both the fiber orientation and the relative crack length. The fiber angle at which the mode mixity is maximum shifts to the higher angles as the relative crack length increases.

The other example we investigated is a rectangular plate with double edge cracks. The effect of anisotropy on the Mode I stress intensity factor was seen to be noteworthy at fiber angles 30 to 60 degrees, as in the single edge crack case.

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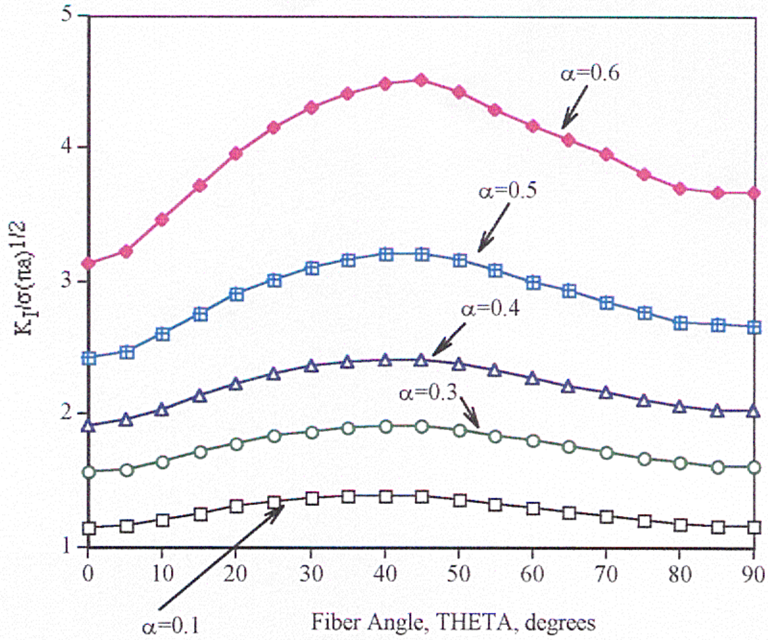

Figure 1a. The effect of anisotropy on the Mode I stress intensity factor of a single-edge crack in a strip under uniform tension; unidirectional graphite/epoxy with fiber orientation, θ, measured from the direction of the applied load

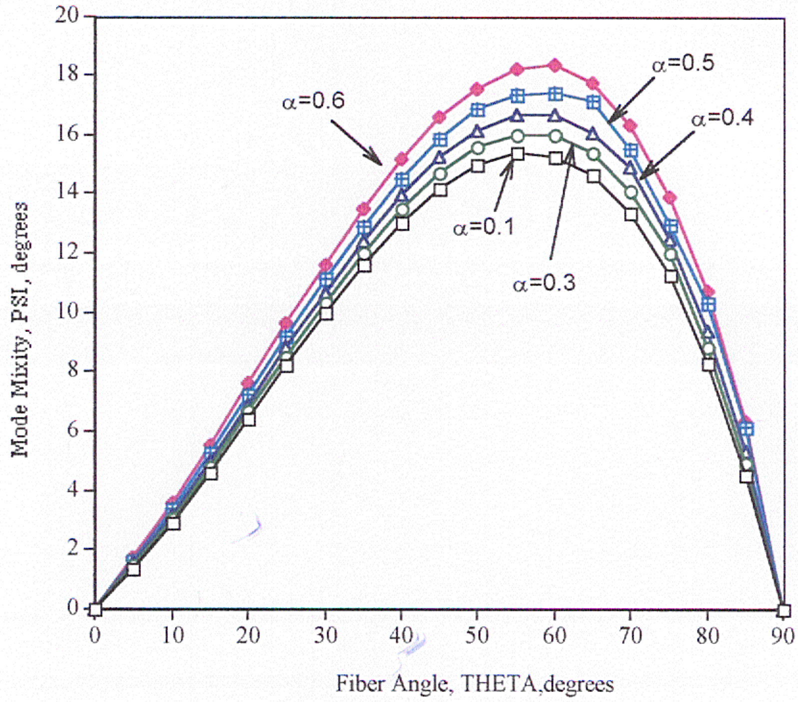

Figure 1b. The effect of anisotropy on the Mode Mixity, ψ, of a single-edge crack in a strip under uniform tension.