IMPROVEMENT OF UNIDIRECTIONAL COMPOSITE STRENGTH PROBABILISTIC MODEL

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SUMMARY: This article presents improvements on unidirectional composite strength probabilistic model. This paper shows how to increase the preciseness of simulation, first by using a simple mechanical model to predict the load transfer around fibre breaks, and secondly by developing the probabilistic equations at the second order. From experimental multifragmentation test results and due to these improvements, the model is able to give the mean of the composite strength with an error of 8 % instead of 40 %. Nevertheless the model which is dedicated to predict also the strength scattering, underestimates it highly. That asks about the real origin of the scattering of unidirectional strength, underlining the testing problems and the design rules which take into account the reliability and the scale effect.

KEYWORDS: Unidirectional, load transfer, strength probabilistic model, mean, standard deviation, ineffective length.

INTRODUCTION

The unidirectional composite strength presents important scattering that must be taken into account by engineers to design reliable structures. For that, experimental testing by normalised process is important. Nevertheless, it is also interesting to have models allowing the simulation of composites properties in function of the constituents, in order to improve their synergy. The reason of this high scattering is always under the scope of scientists [1, 2, 3, 4, 5, 6, 7]. However, rare are the articles that present a global approach from the experimental determination of the fibre defects density to a probabilistic model allowing the strength prediction with a micro-mechanical model to determinate the load transfer around broken fibres. So that if many models find very precisely the experimental values of the strength scattering, a hard point is the origin of the experimental values. In this article, we propose the development of both micro-mechanical [8, 9, 10, 11] and probabilistic [6] models to evaluate this kind of modelisation. Moreover, the input parameters linked with the fibre strength are determined from experimental multifragmentation results followed by acoustic waves [12]. By this way, we show the importance of the various parameters of this modelisation and the weak scattering predicted by a complete approach that implies the great role of the testing process in the measurement of the strength variation.

IMPROVEMENT OF THE MICROMECHANICAL MODEL

The probabilistic models of unidirectional composite strength are based on the evaluation of the probability of a crack growth. This probability is not only linked with the stress concentration in the crack plan but also with all over-stressed length of the fibres. These both phenomena are correlated and named: stress concentration factor K_i and ineffective length (or transfer length) δ_i (Fig. 1), where i is the number of broken fibres in the crack. The K_i are related with the critical crack size and represent the influence of fibre/matrix interface on the composite strength. δ_i measures the capability of matrix to transfer longitudinally the stress. Beyond this length, the broken fibres are considered to be reloaded at the mean stress. Due to the exponential form of the load transfer, the percentage of load (α= 90%, 95%, 99%) for which the fibre is considered to be reloaded, determines the value of δ_i. In the absolute, their determination necessitates a pluridisciplinary study (chemistry, tribology, mechanic). Nevertheless the experimental evaluation of these parameters is difficult due to the size and the number of components. They are statistical data but the problem complexity leads us to consider them as deterministic data.

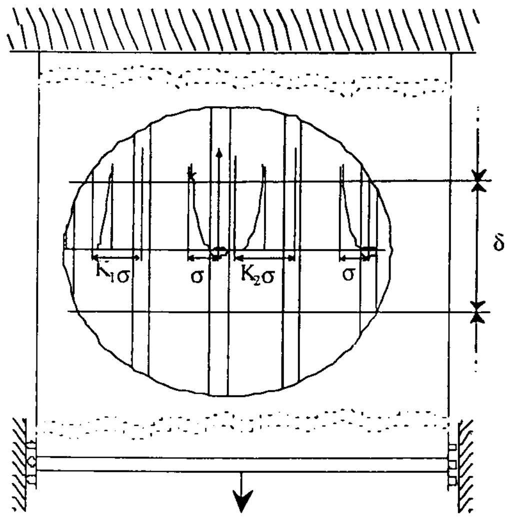

Fig. 1: *ineffective length δ_i and stress concentration factor K_i around broken fibres*

Modelisation

The values of δ_i and K_i depend on the micromechanical modelisation of the broken fibres environment. The various models are differentiated by the estimated behaviour of the matrix: elastic, elasto-plastic; of the interface: cohesion, decohesion; and by the evaluation mode. To reduce the difficulties, most modelisations cut the problem in two independent parts: δ_i and K_i. Some are limited to composite band that over-estimates highly the value of K_i. Moreover the studies are often reduced to regular geometry of crack; that is not practical in the use of probabilistic models. Gao *et al.* [11] have created a model that we have studied and developed.

The composite is considered as a regular arrangement of circular fibres with a radius r_f (square, hexagonal, in staggered rows). The distance between the fibres is equal to d. To evaluate δ_i and K_i, we simplify the geometry of a cracked composite. To treat the problem in a cylindrical co-ordinates system where the stress is simply expressed, the crack and the composite are represented by concentric rings (Fig. 2).

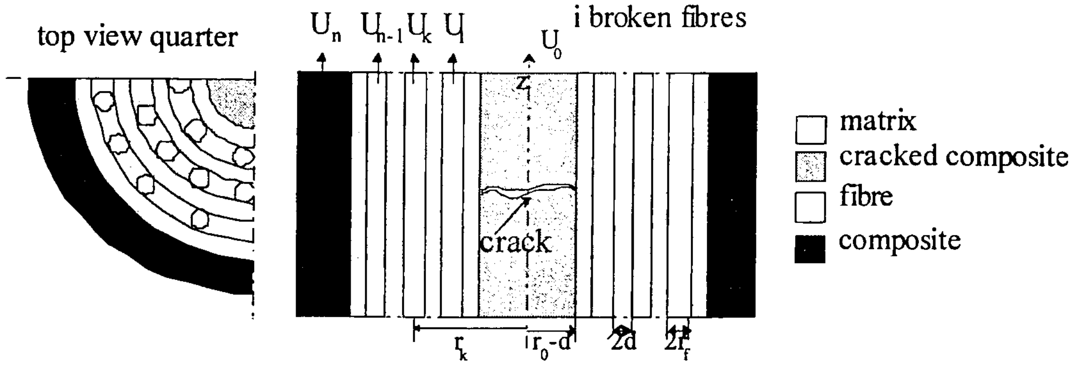

Fig 2 : *Modelisation of the microstructure*

The central disc with a radius r_0 represents the i broken fibres. It is considered to be a r_0-d disk of a homogeneous material surrounded by a matrix ring with a thickness d. r_0 is obtained by considering the whole space of the i broken fibres. The mixing law gives the Young's modulus E of the homogenised material by using the proportion between matrix and fibres :

$$\pi r_0^2 = i(A_f + A_m) \text{ and } \pi(r_0 - d)^2 E = iA_f E_m + \left[iA_m - \pi(r_0^2 - (r_0 - d)^2)\right]E_m \qquad (1)$$

f and m indexes represent fibre and matrix; A is the area occupied by the component, matrix or fibre, in the basic microstructure (square, hexagonal...); i.e. A_f is the fibre section; E are Young's modulus. Next the rings are alternatively matrix ring with a 2d thickness and fibre ring with a $2r_f$ thickness. The last ring is a homogenised material uniformly stressed.

The evaluation of δ_i and K_i is based on shear-lag analysis [1]. Due to the rigidity difference between matrix and fibre, the matrix is only shear-stressed; the matrix equilibrium, the Hooke law and the boundary condition lead to an equation linking the displacements, for U_{i-1} and U_i :

$$\frac{\partial \tau_{Mrz}}{\partial r} + \frac{\tau_{Mrz}}{r} = 0 \Rightarrow \tau_{Mrz} = \frac{cst}{r} = G_m \frac{\partial U_{Mrz}}{\partial r} \Rightarrow (U_1 - U_0) = \frac{cst}{G_m} \ln\left(\frac{r_0 - d}{r_0 + d}\right) \quad \text{for i = 1} \qquad (2)$$

where G_m is the shear modulus of the matrix. Forces equilibrium, the Hooke law in the fibre rings give another equations; for i=1 for example:

$$\frac{dP_f}{dz} = \frac{d}{dz}\left(E\pi(r_0 - d)^2 \frac{dU_0}{dz}\right) = 2\pi(r_0 - d)\tau_{frz} = 2\pi cst \qquad (3)$$

Combining the equations (2) and (3), we obtain :

$$E\pi(r_0 - d)^2 \frac{d^2U_0}{dz^2} + 2\pi G_m(U_1 - U_0)/\ln\left(\frac{r_0 - d}{r_0 + d}\right) = 0 \qquad (4)$$

The same equilibrium equations can be written for each rings; for the k^{th} ring, the radial position r_k is equal to $r_0+(2k-1)(r_f+d)$. For geometrical reasons, the number n_{ik} of fibres in the k^{th} ring, is approached by the integer part of $(\pi/\arcsin((r_f+d)/r_k))$ that represents the fibres number entering in the circumference. The equation corresponding to the k^{th} ring is:

$$n_{ik}A_f E_f \frac{d^2U_k}{dz^2} + 2\pi G_m\left[\frac{(U_{k+1} - U_k)}{\ln((r_k + r_f + 2d)/(r_k + r_f))} - \frac{(U_k - U_{k-1})}{\ln((r_k - r_f)/(r_k - r_f - 2d))}\right] = 0 \qquad (5)$$

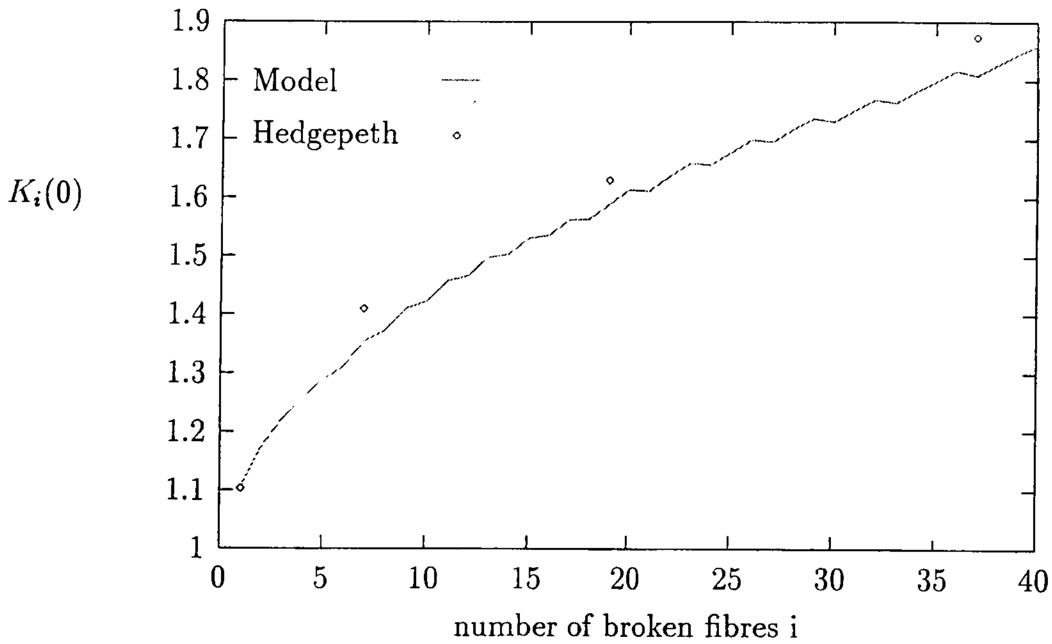

Fig. 3: *Coefficient of stress concentration versus the broken fibre number compared with Hedgepeth's results [9]*

It leads to a matrix system; {U} is the displacement vector ($\{U_0,U_1,...,U_{n-1}\}^t$):

$$\frac{d^2}{dz^2}\{U\}+[A]\{U\}=\{C_l\} \qquad \text{with } \{C_l\}^t=\{0,\ 0,...,f^*U_n\} \tag{6}$$

$\{C_l\}$ translates the boundary conditions; the tri-diagonal nxn matrix [A] and f are given:

$$f=\frac{2\pi G_m}{n_{in-1}A_fE_f\ln\left((r_{n-1}+r_f+2d)/(r_{n-1}+r_f)\right)},\ A_{11}=-A_{12}=\frac{-2G_m}{E(r_0-d)^2\ln((r_0+d)/(r_0-d))},...$$

$$A_{k+1k}=\frac{2\pi G_m}{n_{ik}A_fE_f\ln((r_k-r_f)/(r_k-r_f-2d))},\ A_{k+1k+2}=\frac{2\pi G_m}{n_{ik}A_fE_f\ln((r_k+r_f+2d)/(r_k+r_f))}$$

$$A_{nn}=\frac{2\pi G_m}{n_{in-1}A_fE_f}\left(\frac{1}{\ln((r_{n-1}-r_f)/(r_{n-1}-r_f-2d))}+\frac{1}{\ln((r_{n-1}+r_f+2d)/(r_{n-1}+r_f))}\right),\ A_{k+1k+1}=-\left(A_{k+1k}+A_{k+1k+2}\right)$$

The solution of this system necessitates the evaluation of the n eigenvalues Λ_k and vectors $\{X^k\}$ by resolving $([A]+\Lambda\,[I])\{X\}=0$. The particular solution is given by the displacement values far from the crack, $U_0=U_1=...=U_m$ and P_{f0} is the mean fibre loading Then, it comes:

$$\{U\}=\sum_{k=1}^{n}B_k\exp\left(-\sqrt{\Lambda_k}z\right)\left\{X^k\right\}+\frac{P_{f0}z}{A_fE_f}\{I\} \tag{7}$$

The B_k values are given by the boundary condition in the crack plane: the stress is equal to zero in the broken fibres and the displacement is null for the other, then:

$$\sum_{k=1}^{n}\left(B_k-\sqrt{\Lambda_k}\right)X_l^k+\frac{P_{f0}}{A_fE_f}=0 \ \text{ and } \ \sum_{k=1}^{m}B_kX_l^k=0 \text{ for } l\in\{1,2,...,n-1\} \tag{8}$$

This nxn system is then solved. By deriving the displacement, the load of along the fibres are found and immediately translated into δ_i and K_i:

$$\left\{P_f(z)\right\}=A_fE_f\sum_{k=1}^{n}B_k\sqrt{\lambda_k}\exp(-\sqrt{\lambda_k}z)\left\{X_k\right\}+P_{f0}\{I\} \tag{9}$$

$$\{K_i(z)\} = \frac{1}{P_{f0}}\{P_f(z)\} \text{ and } \delta \text{ is such that } \frac{1}{P_{f0}}\{P_f(\delta/2)\}_1 = 0{,}9 \qquad (10)$$

In this work, the data correspond to a glass/epoxy composite. They have been measured in the laboratory or given by the supplier: ν_m=0.377, r_f=12 µm, V_f=0.6, E_f=69 GPa, E_m=4.17 GPa.

As shown in Fig. 3, the results of the model are close to Hedgepeth's ones.

The advantage of this model is that instead of having two parameters to describe the load transfer around the break, we have complete profile of the stress along the fibre. This information may be used in an adapted probabilistic model.

PROBABILISTIC MODELS OF UNIDIRECTIONAL COMPOSITE STRENGTH

There are many probabilistic models to predict the strength of a composite material. Mathematical models [3] have allowed a best understanding of the phenomenon. But they are generally considered unusable. Moreover their use leads to inappropriate approximations. Then a choice must be made between *critical crack* models and random simulation. The last is often used because of its high adaptability. Nevertheless these methods do not take into account the important variation of the ineffective length during the crack propagation. Moreover it is necessary to compute K_i for each step of the crack growth; of course, it is possible to share the loading between the intact fibres as Harlow and Phoenix, but it overestimates $K_1(0)$ at least of 78%. Another possibility is to use finite element method to evaluate the stress after each fibre break in the composite but this solution is limited by the complexity of the microstructure. The few ones who have used this solution have studied only 2D composites without reaching the convergence of the method.

So we are interested in the *critical crack* model of Deng (6). It is admittedly impossible to account for the crack shape, the effect of the composite border and the break by joining two cracks. By hypothesis, these phenomena are considered to be negligible.

Resolution of the Probabilistic Model

The method is based on the probability p_k that there is in the composite a crack joining k fibres in a band of a length δ_k. Under the hypothesis that the fibre strength is a Weibull or sigmoid law, the probability that a fibre breaks in such composite is:

$$p_1(\sigma) = F_{\delta_1}(\sigma) = 1 - \exp(-\delta_1 f(\sigma)) \qquad (11)$$

For a Weibull law: $$f(\sigma) = \sum_{i=1}^{n} (\sigma/\sigma_i)^{\rho_i},$$

for a sigmoid law $$f(\sigma) = A[1 - \exp(\sum_{i=1}^{n} (\sigma/\sigma_i)^{\rho_i})].$$

where n is the number of modes. The probability that a fiber, adjacent to the broken fiber, which has resisted to σ, breaks under the new stress concentration $K_1(z)\sigma$ is equal to

$$p(\sigma) = 1 - \exp\left(-\int_{-\delta_1/2}^{\delta_1/2} f(K_1(z)\sigma)dz\right) - (1 - \exp(-\delta_1 f(\sigma))) \qquad (12)$$

If there is μ_1 adjacent fibres, the probability that the first break leads a 2 fibres crack is :

$$p_{2/1}(\sigma) = \mu_1\left[\exp(-\delta_1 f(\sigma)) - \exp(-I_1(\sigma))\right] \tag{13}$$

where $I_i(\sigma) = 2\int_0^{\delta/2} f(K_i(z)\sigma)dz$. It is possible to generalise the expression of the probability of the growth from an i fibres crack to i+1 fibres crack. One adjacent fibre breaks so μ_{i-1}-1 that resisted to the previous stress concentration resist to the new one; there were also (μ_i+1-μ_{i-1}) that become adjacent and pass from a stress σ to a stress concentration $K_i(z)\sigma$. Then the conditional probability of growth is:

$$\begin{aligned} p_{i+1/i} &= (\mu_i - \mu_{1-1} + 1)\left[\exp(-\delta_i f(\sigma)) - \exp(-I_i(\sigma))\right] \\ &+ (\mu_{1-1} - 1)\left[\exp(-I_{i-1}(\sigma)) - \exp(-I_i(\sigma))\right] \end{aligned} \tag{14}$$

The probability that a crack of k fibres exists in the short composite is equal to the probability that a first crack appears, multiplied by the probabilities of growth from the first crack to the k fibres crack. Then it gives:

$$p_k(\sigma) = F_{\delta_1}(\sigma)\prod_{i=1}^{k-1} p_{i+1/i} \tag{15}$$

The failure probability of the material is associated to the probability of the existence of a critical crack of k* fibres. After this value, the probability of growth $p_{k*/k*-1}$ is upper than 1 and then the growth is unstable; k* may be estimated by:

$$k^* \text{ such that } \mu_{k^*-1} < (1 - \exp(-I_{k^*-1}\sigma))^{-1} \text{ and } \mu_{k^*} > (1 - \exp(-I_{k^*}\sigma))^{-1} \tag{16}$$

If n is the number of the fibres in the composite, L the total length, the maximal number of critical crack that can exist, is:

$$n' = nL / \left[(k^* - 1)\delta_{k^*-1}\right] \tag{17}$$

Due to the law of the weakest link, the failure probability of the whole composite is:

$$P = 1 - (1 - p_{k^*})^{n'} \tag{18}$$

RESULTS OF THE IMPROVEMENTS

The originality of this development is due to the second member of the conditional probability in the equation 12 and to the integration of the stress shape along the fibre which is possible because of the micromechanical model. These two developments lead to a model independent of the choice of the percentage of load for which the broken fibres are considered to be re-loaded (α= 90%, 95%, 99%). In classical model with variation of α between 0,5 and 0.99, the estimation of the strength mean varies of 23%; for the same range of variation of α, the results of the improved method vary only of 2%. This simplifies the researcher 's choice about the tuning parameter α.

Improvement due to the integration

The integration of the stress $K_i(z)\sigma$ instead of the use of a constant over-stress $\max(K_i(z))\sigma$ along the length δ_i leads to a better evaluation of the mean strength of 30%. Some authors use a linear approximation of $K_i(z)\sigma$ equal to $\max(K_i(z))\sigma$ in the crack plane and zero at the distance δ_i; this model correspond to a plastic behaviour of the matrix, it gives a mean strength 18% lower than the presented model. But under the elastic hypothesis, the $K_i(z)\sigma$ is the sum of highly non-linear decreasing exponential functions (cf. eq. 10). The difference is really increased because of the function f, which is composed of power functions in which the powers are between 2.5 and 11. Moreover, various measurements of the stress after a fibre break have shown an exponential shape.

Results of the model

We compare now the results given by the classical development but with the integration of stress and the improved one. The distribution of the defects along the fibres that gives the function $f(\sigma)$ (cf eq. 12) comes from multifragmentation test on a glass fibre E. The Weibull and sigmoid parameters are estimated from this result. The law parameters are presented in the table 1. The results are typically bimodal [12], then contrary to the others, the parameters of the unimodal law are estimated only on the most critical defects. The third mode of the trimodal law is only a mathematical artifice to correct the law; it is subtracted to the other modes of $f(\sigma)$. As it is shown by the remainder R^2, the experimental curve is better fitted by the trimodal laws.

law	σ_1	ρ_1	σ_2	ρ_2	σ_3	ρ_3	A	R^2
Weib. bi	2487	3.49	2014	7.66				0.979
Weib. tri	3600	2.48	2050	9.21	2225	7		0.9995
sig. tri	15025	2.49	2950	10.14	3880	7	34,6	0.9998
Weib. uni	3600	2.48	→ estimated on the most critical defects					0.9247

Tab. 1. : *Parameters of the probabilistic laws of the fibre defect distribution extracted from multifragmentation tests (Weib.: Weibull, sig.: sigmoid, uni, bi, tri : uni-, bi-, tri-modal)*

The models give curves presented in the figure 4; These curves are treated as a weibull law to extract the various parameters that lead to the estimation of the mean and the standard deviation.

The model results are compared with experimental results of traction tests on 48 standard coupons of a unidirectional composite (glass E, epoxy); the experimental law of the strength is a Weibull one with a mean $E[\sigma_c]$=1214 MPa and a coefficient of variation CV=4.96%.

The various estimations of the mean using the classical model show that such modelisation gives only an evaluation of the magnitude of the strength of the composite. Differences with experimental result around 20 and 30% are too important to evaluate efficiently the value of strength. In the other hand, the improved model (emphasized by* in the table 2) evaluate the strength with only 8% of error. This demonstrates the importance of the second order development. Moreover numerical tests have given an idea of the errors made on the $f(\sigma)$ parameters for the various laws - around 5%. Reported into the model, they lead to an interval of variation equal to 10% on the estimation of the strength; then the error found in the estimation of the mean may be explained by the inaccuracy of the input data.

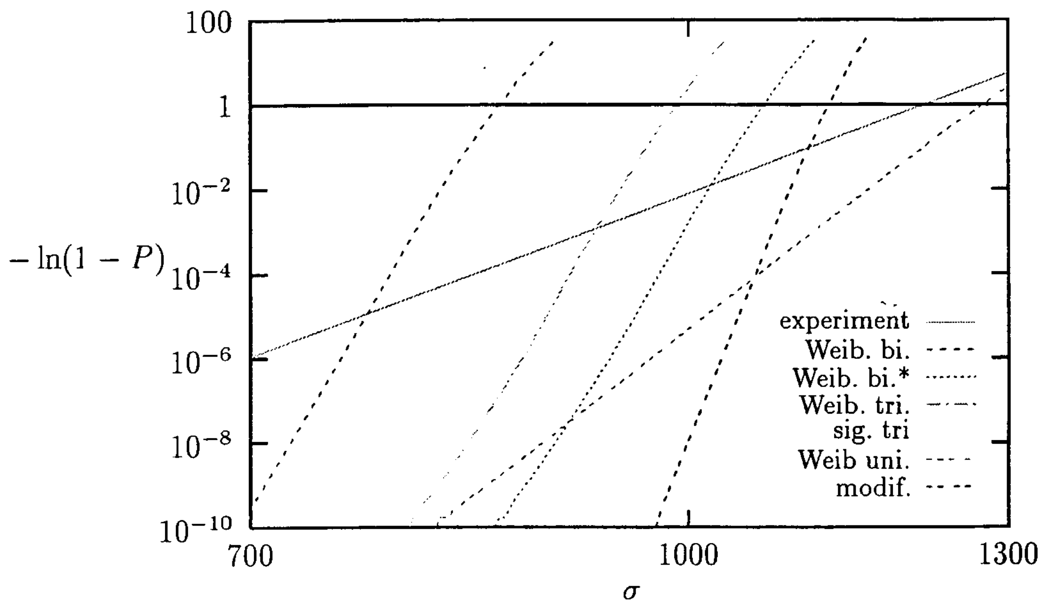

Fig. 4: *Distribution laws of the composite given by the classical and improved methods compared with experimental results*

	Weib. bi	Weib. tri	Weib. bi*	Weib. tri*	sig. tri*	Weib. uni
$E[\sigma_c]$ (MPa)	856	989	1079	1125	1126	1273
Error on E	29,3%	18,1%	11%	8%	8%	5,2%
CV (%)	1,31	1,25	1,25	1,14	1,14	2,96
Error on CV	76,0%	77,3%	77,3%	79,2%	79,2%	46,2%
k*	14	13	17	17	17	17

Tab. 2. : *Mean of strength and coefficient of variation, k* length of the critical crack and difference between the experimental values and the classical and improved(*) models*

In the other hand, the evaluation of the coefficient of variation is difficult. Both models are completely unable to estimate the magnitude of the coefficient of variation of the composite. The factor between the experimental value and the numerical one is close to 5. It is true that the model is not yet developed to take into account all the stochastic variables of such material-structures. For example, the Young's modulus that is considered as a deterministic data, has a coefficient of variation around 3%. This leads to variation of the elastic parameters, which are perhaps not negligible in the evaluation of K_i. Then we can only conclude that the model attributes to the fibres strength variation only a coefficient of variation equal to 1% for composite strength. Then the experimental variation is certainly due to other micromechanical variations: Young's modulus, the fibres ratio, the fibre arrangement, the debounding around fibre break... but above all, the variation is due to experimental problem of coupons manufacturing and testing. It is obviously impossible to evaluate the variation of experimental testing by only stochastic variation of the material components properties. Then it signifies that theoretically the variation of composite strength may be highly reduced more by improving the manufacturing and testing process than by improving the fibre quality.

Influences of the law

The Weibull and sigmoid laws give very close results. This study demonstrates that it is pointless to consider a sigmoid distribution for fibre defects despite the conclusion of

Baxevanakis [13]. This law must be here avoided because of its complexity which implies the numerical integration of $I_i(\sigma)$; that multiplies the computing cost for a negligible gain of preciseness. Therefore the introduction of this law has the merit to break the constant use of Weibull law in such problems. It is impossible to advice a kind of law more than an other. Simply, the laws that better fit the experimental law of fibre defects give the best results.

It is important to note that a classical model using an incomplete data as a unimodal Weibull law estimated only with the most critical defects, gives the best approximation of the experimental results: 5% of difference on the mean and an error divided by 2 for the coefficient of variation. Then an estimation of the mean and the coefficient of variation of the composite strength with limited input data may lead to other conclusions. Then it proves that it is important to estimate the whole distribution of fibre defects using multifragmentation tests or other testing methods in accordance with the fibre material.

CONCLUSION

We have compared experimental results to the evaluation given by a probabilistic model of unidirectional composite strength for which the input data come from multifragmentation test. The classical probabilistic model has been improved by using a micromechanical modelisation of the composite allowing the evaluation of the longitudinal stress along the fibres adjacent to a crack. The stress has been integrated in the model. Next the classical model has been developed at a higher degree of probability; that leads to improve the result of 10% and to make it independent from a tuning parameter. Finally The estimation of the mean given by the improved model is close to the experimental results with an error of 8% instead of 40% for the classical one without any improvement. Moreover the difference between numerical and experimental result may be explained by the uncertainties on the input data.

On the other hand the coefficient of variation is highly underestimated by the model that does not account the experimental uncertainties. The result shows that the fibre defects are perhaps not responsible of the main part of variation in composite strength but the testing techniques are certainly the real cause. That means that the scale effect is probably also a problem of experimentation.

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