# A FLAT SHELL COMPOSITE ELEMENT INCLUDING PIEZOELECTRIC ACTUATORS

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**SUMMARY**

A four-node, twenty-four degrees of freedom flat shell finite element for the analysis of thin composite laminate structures with surface bonded piezoelectric layers is presented. The formulation includes the effect of actuation strain on a plane stress functional containing drilling degrees of freedom. Inplane displacements are interpolated according to a modern membrane element and bending degrees of freedom are represented by the DKQ element.

**KEYWORDS**: composite shell, shell finite element, smart structures

## INTRODUCTION

Several models for the analysis of composite material structures equipped with layers of piezoelectric material have been proposed. Crawley and de Luis [1] developed analytic and numeric solutions using the Rayleigh-Ritz method for the analysis of Euler-Bernoulli beams. Crawley and Lazarus [2] presented Rayleigh-Ritz formulations including the piezoelectric effect in the classical laminate theory.

Earlier the finite element method was used by Allik e Hughes [3] to develop a tetrahedral element with twelve displacement degrees of freedom and four electric degrees of freedom for the analysis of laminated plates with piezoelectric laminae. Tzou and Tseng[4] and Ha, Keilers and Chang [5] adapted a tridimensional hexaedron with twenty four displacement degrees of freedom and eight electrical degrees of freedom to model plates and shells. Finite element solutions based on the Mindlin-Reissner theory including piezoelectric effects were formulated by Joshi and Chan [6], with a quadrilateral element with nine nodes and forty five displacement degrees of freedom. Hwang and Park [7] used the DKQ element [8] to model thin plates, for the solution of vibration control of laminated plates with piezoelectric layers. Zhou, Lai, Xue, Huang and Mei [9] investigated the optimal positioning of piezoelectric actuators for flutter suppression in isotropic plates using a rectangular plate finite element with twenty four degrees of freedom and $C^1$ continuity. In a similar work, Nam and Kim [10] present a solution for flutter control of plates.

In this work it is our intention to present a formulation of an efficient and simple quadrilateral flat shell finite element, with six displacement degrees of freedom per node, for the analysis of thin composite laminate plates and shells equipped with piezoelectric actuators.

Flat elements are used with advantage to model composite shells, due to the excellent results produced and the easy with which they can mix with other types of finite elements.

The finite element is based on the conjunction of a modern membrane quadrilateral finite element with drilling degrees of freedom and the classical DKQ (*discrete Kirchhoff quadrilateral*). Its variational formulation is carried out with the introduction of the piezoelectric effects in a functional which includes an independent field of rotations normal to the plane.

## VARIATIONAL FORMULATION

The model is developed from the following functional originally proposed by Reissner [11] and modified by Hughes and Brezzi [12],

$$\Pi = \int_V \int_\varepsilon \boldsymbol{\sigma}^{\mathrm{T}}\, d\boldsymbol{\varepsilon}\, dV + \int_A \tau\left(\omega_{xy} - \theta_z\right) dA - \tfrac{1}{2}\gamma^{-1}\int_A \tau^2\, dA\,. \tag{1}$$

The first term corresponds to the strain energy and the second to the relaxation from the constraint of equality between independent rotations normal to the plane $\theta_z$ and the antisymmetric components of the displacement gradient $\omega_{xy} = v_{,x} - u_{,y}$, where $\tau$ is the Lagrange multiplier. Finally, the third term is included to allow static condensation for elimination of $\tau$ from Eqn 1, where $\gamma$ is a coefficient.

The deformations are given by

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\circ} + z\boldsymbol{\kappa} \tag{2}$$

where accordingly to the theory of thin plates and shells,

$$\boldsymbol{\varepsilon}^{\circ} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix} \tag{3}$$

are the midsurface inplane strains and

$$\boldsymbol{\kappa} = \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{Bmatrix}. \tag{4}$$

are the midsurface curvatures.

The constitutive relation includes the piezoelectric effect as an initial deformation, such that

$$\boldsymbol{\sigma} = \overline{\mathbf{Q}}\{\boldsymbol{\varepsilon} - \boldsymbol{\Lambda}\} \tag{5}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are respectively the stress and strain tensors for plane stresses state, $\overline{\mathbf{Q}}$ is the laminate stiffness matrix. The piezoelectric inplane deformations are represented by [13],

$$\boldsymbol{\Lambda} = \mathbf{d}_0\, E_3 + \boldsymbol{\chi}\, E_3^{\,2} + \boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{\varepsilon}\, E_3 . \tag{6}$$

$E_3$ is the electric field normal to the midsurface and $\mathbf{d}_0$, $\boldsymbol{\chi}$ and $\boldsymbol{\psi}$, are the coefficients of electromechanical coupling. In Eqn 6 the dependency between piezoelectric deformation $\boldsymbol{\Lambda}$ and elastic strain $\boldsymbol{\varepsilon}$ is apparent, as verified experimentally in [2]. Substitution of Eqn 6 in Eqn 5 leads to

$$\boldsymbol{\sigma} = \overline{\mathbf{Q}}\left[\mathbf{I} - \boldsymbol{\psi}\, E_3\right]\boldsymbol{\varepsilon} - \overline{\mathbf{Q}}\left\{\mathbf{d}_0\, E_3 + \boldsymbol{\chi}\, E_3^{\,2}\right\}. \tag{7}$$

The integration of Eqn 1 after the substitution of Eqns 7 and 2 results

$$\Pi = \tfrac{1}{2}\int_A \begin{Bmatrix} \boldsymbol{\varepsilon}^{\circ} \\ \boldsymbol{\kappa} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^{\circ} \\ \boldsymbol{\kappa} \end{Bmatrix} dA - \int_A \begin{Bmatrix} \mathbf{N}_{\Lambda} \\ \mathbf{M}_{\Lambda} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \boldsymbol{\varepsilon}^{\circ} \\ \boldsymbol{\kappa} \end{Bmatrix} dA + \int_A \tau\left(\omega_{xy} - \theta_z\right) dA - \tfrac{1}{2}\gamma^{-1}\int_A \tau^2\, dA \tag{8}$$

The matrices **A**, **B** and **D** are given by

$$(\mathbf{A},\mathbf{B},\mathbf{D}) = \int_t \left[\mathbf{I} - \boldsymbol{\psi} E_3\right]^{\mathrm{T}} \overline{\mathbf{Q}}\ \ (1, z, z^2)\ \ dz \tag{9}$$

and

$$(\mathbf{N}_{\Lambda}, \mathbf{M}_{\Lambda}) = \int_t \overline{\mathbf{Q}}\left\{\mathbf{d}_{\circ} E_3 + \boldsymbol{\chi} E_3^{\,2}\right\}\ (1, z)\ \ dz \tag{10}$$

are the piezoelectric force and moment resultants along the thickness.

## FINITE ELEMENT SOLUTION

A 4-node finite element is proposed, as shown in Fig.1, with the following membrane nodal displacement degrees of freedom,

$$\mathbf{u}_m^{\mathrm{T}} = \left\{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4\right\}, \tag{11}$$

drilling degrees of freedom,

$$\boldsymbol{\theta}_z^{\mathrm{T}} = \left\{\theta_{z1} \quad \theta_{z2} \quad \theta_{z3} \quad \theta_{z4}\right\}, \tag{12}$$

and bending displacement degrees of freedom,

$$\mathbf{u}_p^{\mathrm{T}} = \left\{w_1 \quad \theta_{x1} \quad \theta_{y1} \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \quad w_3 \quad \theta_{x3} \quad \theta_{y3} \quad w_4 \quad \theta_{x4} \quad \theta_{y4}\right\}. \tag{13}$$

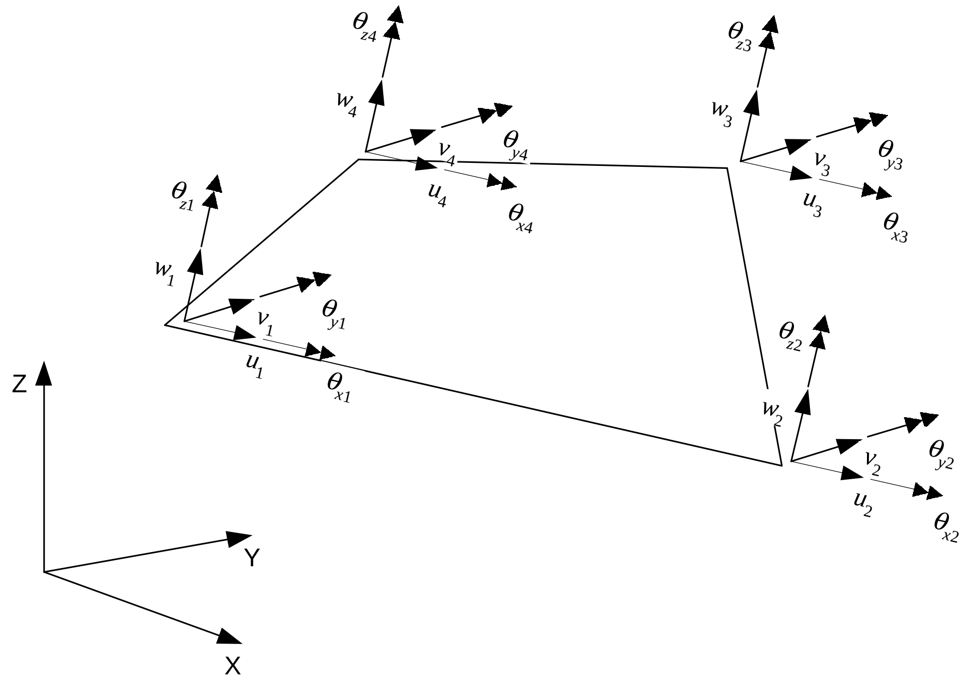


Fig. 1. *Flat shell element with nodal displacement dofs*

According to the formulation by Ibrahimbegovic, Taylor and Wilson [14] the inplane strains are interpolated by

$$\varepsilon^{\circ} \cong \mathbf{B}_m \mathbf{u}_m + \mathbf{G}\boldsymbol{\theta}_z \tag{14}$$

while the terms corresponding to rotations normal to the plane are interpolated according to

$$\omega_{xy} - \theta_z \cong \mathbf{b}\mathbf{u}_m + \mathbf{g}\boldsymbol{\theta}_z\,. \tag{15}$$

Expressions for the matrices **B**, **G**, **b** and **g** are found in [14].

The curvatures are expressed according to the DKQ element of Batoz and Tahar [8]

$$\kappa \cong \mathbf{B}_p \mathbf{u}_p\,. \tag{16}$$

The Lagrange multiplier $\tau$ is considered constant over the element [14], that is,

$$\tau \cong \tau_0 \tag{17}$$

The substitution of Eqns 14-17 in the functional of Eqn 8 results, after variation,

$$
\begin{aligned}
\delta\Pi = \sum_e \delta \begin{Bmatrix} \mathbf{u}_m \\ \boldsymbol{\theta}_z \\ -- \\ \mathbf{u}_p \\ -- \\ \tau_0 \end{Bmatrix}^{\mathrm{T}} & \left( \int_{A^e} \left[ \begin{array}{c:c:c} [\mathbf{B}_m\mathbf{G}]^{\mathrm{T}}\mathbf{A}[\mathbf{B}_m\mathbf{G}] & [\mathbf{B}_m\mathbf{G}]^{\mathrm{T}}\mathbf{B}\mathbf{B}_p & \langle\mathbf{bg}\rangle^{\mathrm{T}} \\ \hdashline \mathbf{B}_p^{\mathrm{T}}\mathbf{B}[\mathbf{B}_m\mathbf{G}] & \mathbf{B}_p^{\mathrm{T}}\mathbf{D}\mathbf{B}_p & \mathbf{0} \\ \hdashline \langle\mathbf{bg}\rangle & \mathbf{0} & -\gamma^{-1}A^e \end{array} \right] dA \begin{Bmatrix} \mathbf{u}_m \\ \boldsymbol{\theta}_z \\ -- \\ \mathbf{u}_p \\ -- \\ \tau_0 \end{Bmatrix} \right. \\
& \left. - \int_{A^e} \begin{bmatrix} [\mathbf{B}_m\mathbf{G}] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \mathbf{N}_\Lambda \\ \mathbf{M}_\Lambda \\ 0 \end{Bmatrix} dA \right) = 0
\end{aligned} \tag{18}
$$

The first integral is the characteristic matrix and the second the piezoelectric equivalent load vector. Since the Lagrange multiplier $\tau$ is interpolated independently in each element, its part can be condensed out from the characteristic matrix, resulting

$$
\begin{aligned}
\delta\Pi = \sum_e \delta \begin{Bmatrix} \mathbf{u}_m \\ \boldsymbol{\theta}_z \\ -- \\ \mathbf{u}_p \end{Bmatrix}^{\mathrm{T}} & \left( \int_{A^e} \left[ \begin{array}{c:c} [\mathbf{B}_m\mathbf{G}]^{\mathrm{T}}\mathbf{A}[\mathbf{B}_m\mathbf{G}] + \frac{\gamma}{A^e}\langle\mathbf{bg}\rangle^{T}\langle\mathbf{bg}\rangle & [\mathbf{B}_m\mathbf{G}]^{\mathrm{T}}\mathbf{B}\mathbf{B}_p \\ \hdashline \mathbf{B}_p^{\mathrm{T}}\mathbf{B}[\mathbf{B}_m\mathbf{G}] & \mathbf{B}_p^{\mathrm{T}}\mathbf{D}\mathbf{B}_p \end{array} \right] dA \begin{Bmatrix} \mathbf{u}_m \\ \boldsymbol{\theta}_z \\ -- \\ \mathbf{u}_p \end{Bmatrix} \right. \\
& \left. - \int_{A^e} \begin{bmatrix} [\mathbf{B}_m\mathbf{G}] & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_p \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \mathbf{N}_\Lambda \\ \mathbf{M}_\Lambda \end{Bmatrix} dA \right) = 0
\end{aligned} \tag{19}
$$

The element stiffness matrix is recognized as

$$
\mathbf{K}^e = \int_{A^e} \begin{bmatrix} [\mathbf{B}_m\mathbf{G}]^{\mathrm{T}}\mathbf{A}[\mathbf{B}_m\mathbf{G}] + \frac{\gamma}{A^e}\langle\mathbf{bg}\rangle^{\mathrm{T}}\langle\mathbf{bg}\rangle & [\mathbf{B}_m\mathbf{G}]^{\mathrm{T}}\mathbf{B}\mathbf{B}_p \\ \mathbf{B}_p^{\mathrm{T}}\mathbf{B}[\mathbf{B}_m\mathbf{G}] & \mathbf{B}_p^{\mathrm{T}}\mathbf{D}\mathbf{B}_p \end{bmatrix} dA \tag{20}
$$

while the vector of piezoelectric equivalent load is

$$
\mathbf{F}_{pz}^e = \int_{A^e} \begin{bmatrix} [\mathbf{B}_m\mathbf{G}] & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_p \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \mathbf{N}_\Lambda \\ \mathbf{M}_\Lambda \end{Bmatrix} dA\,. \tag{21}
$$

The results for displacements are barely affected by the value given to the coefficient $\gamma$, which is recommended to be taken equal to the shear modulus of the predominant material in the laminate. For symmetric laminates there is a decoupling between membrane and bending effects resulting in the following stiffness matrix,

$$
\mathbf{K}^e = \int_{A^e} \begin{bmatrix} [\mathbf{B}_m\mathbf{G}]^{\mathrm{T}}\mathbf{A}[\mathbf{B}_m\mathbf{G}] + \frac{\gamma}{A^e}\langle\mathbf{bg}\rangle^{\mathrm{T}}\langle\mathbf{bg}\rangle & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_p^{\mathrm{T}}\mathbf{D}\mathbf{B}_p \end{bmatrix} dA \tag{22}
$$

The stiffness matrix and the vector of piezoelectric equivalent load are obtained trough a 2×2 integration rule.

## NUMERICAL RESULTS

### *Membrane Problem*

The membrane made of one layer of G1195 PZT ceramic is discretized with only one finite element of unit size. Its deformations are calculated for several different voltages, in steps of 25V up to 150V, using the material model of Eqn 6, with the coefficients given in Table 1, extracted from [13]. The results obtained agree precisely with the experimental results reported graphically in [2], which exhibits a slightly nonlinear curve of strain versus voltage.

### *Cylindrical Shell Problem*

The cylindrical shell is 152mm wide, 269mm long, and the radius of the inner surface is 304.8mm. The shell has six layers of T300/976 in a $[90/+60/-60]_s$ construction, covered with a continuous layer of Kynar piezoelectric film in the inner surface only. The angles are measured with respect to the direction coinciding with the *x* axis. Although the piezoelectric film has mechanically isotropic behavior it exhibits an orthotropic voltage versus strain behavior. The shell is clamped in one corner as shown in Fig. 1. The lack of symmetry added to the unusual boundary condition on displacements makes the problem rather difficult. A 6×3 uniform mesh was utilized to analyze the shell for several different applied voltages and the results were used to plot the vertical displacement of point A, as shown in Fig. 2. The results are excellent, agreeing very well with the experimental and numerical data reported in [13]. The solution adopted in [13] includes also transverse shear effects, not considered here. The displacement results obtained for point B are as good as those obtained for point A.

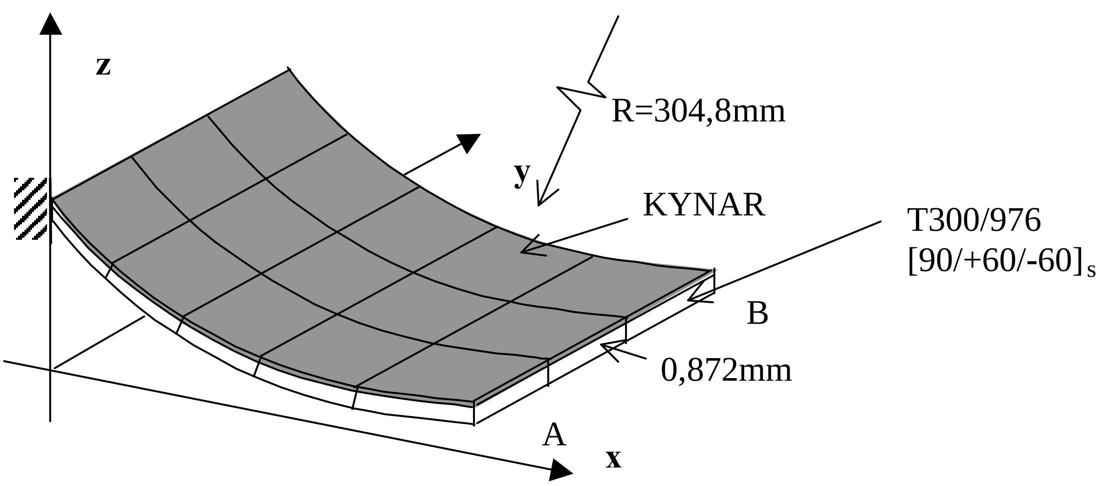


Fig. 1. *Cylindrical shell with one side bonded piezoelectric layer*

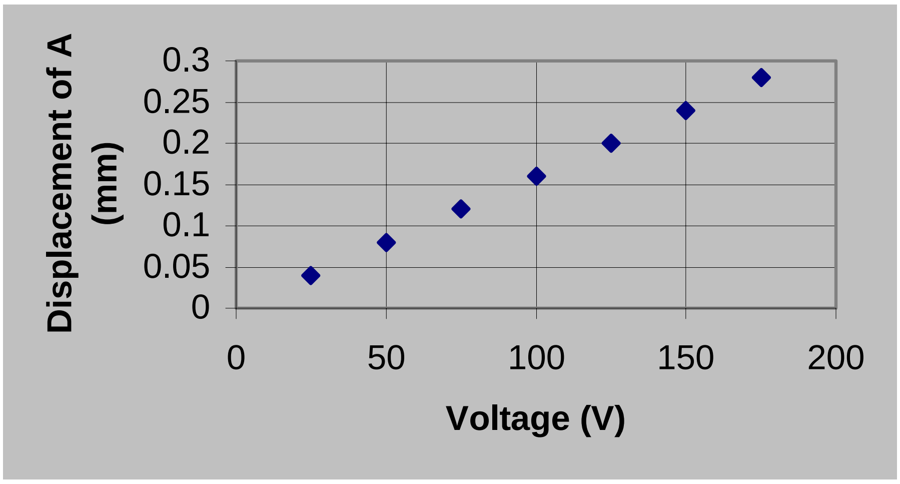


Fig. 2. *Vertical displacement of point A of the cylindrical shell*

Table 1. *Material Properties*

| | Unit | T300/976 | KYNAR | G1195 |
|---|---|---|---|---|
| t | $10^{-3}$ m | 0.127 | 0.110 | 0.254 |
| $E_x$ | $10^{9}$ Pa | 150 | 2 | 63 |
| $E_y$ | $10^{9}$ Pa | 9 | 2 | 63 |
| $G_{xy}$ | $10^{9}$ Pa | 7.1 | 0.77 | 24.2 |
| $\nu_{xy}$ | - | 0.3 | 0.29 | 0.3 |
| $d_{31}$ | $10^{-12}$ m/V | - | 23 | 264 |
| $d_{32}$ | $10^{-12}$ m/V | - | 4.6 | 264 |
| $\chi_{31}$ | $10^{-18}$ $m^2/V^2$ | - | - | 3.41 |
| $\psi_1$ | $10^{-9}$ m/V | - | - | 640 |
| $\psi_2$ | $10^{-9}$ m/V | - | - | -162.6 |

## CONCLUSIONS

The formulation of a flat shell finite element based on a variational principle was concisely presented, for the analysis of thin composite material laminates equipped with layers of piezoelectric material. The resulting element can be applied to any, symmetric or not, thin composite plates and shells. Furthermore, the element is also able to model a more refined constitutive piezoelectric behavior than the usual linear assumption. The element presented excellent results, particularly in the analysis of a composite material cylindrical shell with a piezoelectric layer in just one its faces and a punctual displacement boundary condition. The element seems to be a powerful tool available for the analysis of thin composite laminates with piezoelectric layers.

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