# DESIGN OF LAMINATED PLATES WITH REQUIRED IN-PLANE, COUPLING AND BENDING STIFFNESSES USING NON-STANDARD PLY ANGLES

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**SUMMARY**: The purpose of this paper is to describe an optimization procedure that provides stacking sequences fulfilling required stiffness properties. The elastic properties of the plates to be designed are described by twelve lamination parameters. An objective function involving these twelve required parameters is minimized with respect to the ply orientations that are considered as continuous variables. A method based on the gradient method is used to perform the minimization. Various examples of laminated plates exhibiting particular elastic properties illustrate the approach. In conclusion, all required properties are obtained with twelve plies and the stacking sequences are not symmetric, even though the corresponding plates are uncoupled.

**KEYWORDS**: design, laminated plate, optimization, stiffness.

## INTRODUCTION

The classical lamination theory [1] provides a frame of reference for analyzing the elastic properties of laminated plates. When a constitutive material and a stacking sequence are chosen, one can easily compute the three different stiffness matrices that govern the elastic properties of the plate. The stacking sequences are often determined with some classical rules. For instance, the ply angles are set to some pre-selected ply angles (for instance 0°, 45°, 90°) or the laminated plates are symmetric to obtain a zero coupling matrix. However, it must be pointed out that the inverse problem that consists of the definition of a stacking sequence from required stiffness properties has not been solved yet in the general case. Some analytical solutions are available, but the number of plies is fractional [2] or the minimum total number of plies required to achieve the properties is important, 36 plies for instance in the case of fully isotropic laminates [3]. Numerical methods based on optimization procedures are also available. They are often based on genetic algorithms since the ply orientations are to be set to some pre-selected ply angles in this case [4].
The aim of this work is to provide a numerical approach to define stacking sequences from required elastic properties. These properties are considered to be defined by a set of lamination parameters and the goal is to obtain some stacking sequences that exhibit these

required parameters. The present approach clearly departs from the usual ones available in the literature: the ply angles are considered as continuous and the stacking sequences are not symmetric *a priori* even though uncoupled plates are determined. The main advantage is to give some additional degrees of freedom that will turn out to be useful to design thinner plates than those which would be obtained with the classical rules. Since the angles are presently assumed to be continuous, the numerical procedure used for the optimization is based on the well-known gradient method that has been modified to take into account the problem of the local minima of the objective-function.

The design method and its numerical implementation are presented in the two first parts of the paper. The approach is then illustrated with some examples of thin laminated plates exhibiting particular properties.

## STATEMENT OF THE PROBLEM

### Definition of the lamination parameters

Within the framework of the classical lamination theory, the elastic properties of a laminated plate is defined by three normalized stiffness matrices $A^*$, $B^*$ and $D^*$ that describe respectively the in-plane, coupling and bending behaviours [1]. The calculation of the matrices components can be achieved advantageously by introducing invariant quantities $U_i$ i=1..5, and lamination parameters $V_{i\alpha}$ i=1..4, $\alpha$=A, B, D. For instance, the in-plane stiffnesses may be described as follows

$$\begin{pmatrix} A^*_{11} & A^*_{12} & A^*_{16} \\ A^*_{12} & A^*_{22} & A^*_{26} \\ A^*_{16} & A^*_{26} & A^*_{66} \end{pmatrix} = \begin{pmatrix} U_1 & U_4 & 0 \\ U_4 & U_1 & 0 \\ 0 & 0 & U_5 \end{pmatrix} + U_2 \begin{pmatrix} V_{1A} & 0 & \frac{V_{3A}}{2} \\ 0 & -V_{1A} & \frac{V_{3A}}{2} \\ \frac{V_{3A}}{2} & \frac{V_{3A}}{2} & 0 \end{pmatrix} + U_3 \begin{pmatrix} V_{2A} & -V_{2A} & V_{4A} \\ -V_{2A} & V_{2A} & -V_{4A} \\ V_{4A} & -V_{4A} & -V_{2A} \end{pmatrix} \quad (1)$$

A similar relationship holds for both the $B^*$ and $D^*$ matrices: the subscript is changed into B and D respectively. Moreover, in the case of the $B^*$ matrix, the first matrix in the right hand side built up with the invariant parameters $U_i$ i=1, 4, 5 disappears.

The lamination parameters are defined by

$$\left(V_{1\alpha}, V_{2\alpha}, V_{3\alpha}, V_{4\alpha}\right) = \sum_{i=1}^{n} w_\alpha(i)\left(\cos 2\theta_i, \cos 4\theta_i, \sin 2\theta_i, \sin 4\theta_i\right), \alpha = A, B, D \quad (2)$$

where the $w_\alpha(i)$ are wheighting factors defined for each of the three matrices [1] and n is the number of plies.

If we consider now the problem of finding a set of angles leading to required stiffness properties, it is clear that one must first choose a given constitutive material defined by its $U_i$'s, i=1..5 and second define a stacking sequence leading to the twelve required lamination parameters denoted $\overline{V_{i\alpha}}$ , i=1..4, $\alpha$=A, B, D. This second problem is now addressed.

## Definition of a stacking sequence exhibiting required lamination parameters

The problem of the definition of a stacking sequence defined by a set lamination parameters has been addressed first by Miki in the separate cases of in-plane [5] and bending [6] properties. Fukunaga unified this approach by considering simultaneously in-plane and bending properties [2]. These methods are analytical. They lead to some solutions with twelve plies but the thickness is different from one ply to another. In these cases, the stacking sequences are some mixtures of angle-ply and cross-ply laminates. Another approach is to consider *a priori* that the ply angles can be set to some pre-selected standard angles and to use a suitable numerical strategies to obtain a solution. Some genetic algorithms have been proposed recently in the literature ([4] for instance). However, its not sure that any required lamination parameters can be reached with such a small set of ply angles. Moreover, one could consider that the stacking sequences are not symmetric to increase the freedom in the design since the condition of symmetry is sufficient and not necessary.
The design problem is presently considered as an optimization problem. An objective function is first defined and it is minimized with the well-known gradient method which has been modified because the function exhibits many local minima.

## Objective function and minimization

The objective function is built as the sum of the squared differences between the twelve lamination parameters and the twelve corresponding required values $V_{i\alpha}$ and $\overline{V_{i\alpha}}$

$$\begin{aligned} F(\Theta) = {} & \left(V_{1A} - \overline{V_{1A}}\right)^2 + \left(V_{2A} - \overline{V_{2A}}\right)^2 + \left(V_{3A} - \overline{V_{3A}}\right)^2 + \left(V_{4A} - \overline{V_{4A}}\right)^2 \\ & + \left(V_{1B} - \overline{V_{1B}}\right)^2 + \left(V_{2B} - \overline{V_{2B}}\right)^2 + \left(V_{3B} - \overline{V_{3B}}\right)^2 + \left(V_{4B} - \overline{V_{4B}}\right)^2 \\ & + \left(V_{1D} - \overline{V_{1D}}\right)^2 + \left(V_{2D} - \overline{V_{2D}}\right)^2 + \left(V_{3D} - \overline{V_{3D}}\right)^2 + \left(V_{4D} - \overline{V_{4D}}\right)^2 \end{aligned} \tag{3}$$

where $\Theta$ is the vector of the ply angles. The objective function is zero if the laminated plate exactly exhibits the required properties. Its is close to zero if the laminated plate approximately exhibits the required properties. A classical optimization strategy based on the well-known steepest descent method has been used to minimize the objective function. Since this function exhibits many local minima, the method has been improved as follows. First, the programme is run many times from different starting points chosen randomly. The minima found are collected and sorted. The stacking sequence found at the lowest minimum found is then considered as the starting point of a final run of the programme with refined values of the search vector used in the steepest descent. The numerical aspects of the procedure are discussed in Ref. [7]. Some typical results obtained with this approach will now be examined.

# APPLICATION TO THE DETERMINATION OF SOME LAMINATED PLATES EXHIBITING REQUIRED PROPERTIES

## Design area

This aim of this section is to examine the capabilities of this method in some relevant particular cases of orthotropic plates.

The first question is to determine the allowable values for the required lamination parameters. This problem has been addressed by Miki [5][6], Fukunaga [2] and Grenestedt [8]. When uncoupled laminated plates are considered, the four lamination parameters $\overline{V_{iB}}$ , i=1..4, are zero. When orthotropic plates are considered, four out of the eight remaining lamination parameters are zero in the ortotropy axes: $\overline{V_{3A}} = \overline{V_{4A}} = \overline{V_{3D}} = \overline{V_{4D}} = 0$. The four non-zero remaining parameters are not independent [5][6]. Let us now consider two points defined with these remaining non-zero parameters: $Q_A\left(\overline{V_{1A}}, \overline{V_{2A}}\right)$ and $Q_D\left(\overline{V_{1D}}, \overline{V_{2D}}\right)$. It has been shown [5] [6] that they are located inside a design area of particular shape in the $V_{1\alpha}$ - $V_{2\alpha}$ planes, $\alpha$=A, D, since this area is bounded by a straight line and a parabola (see Fig. 1). Hence, an uncoupled orthotropic plate is defined by these two points that are called design points in the following since the design of a stacking sequence from these two points is addressed in the present work.

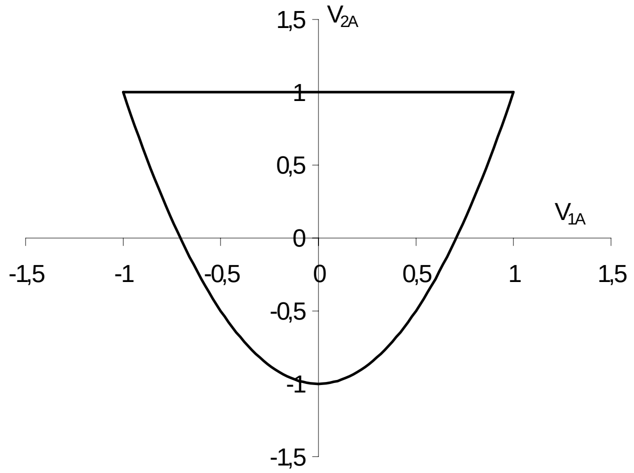



Fig. 1: Design area for the in-plane lamination parameters (the design area for the bending lamination parameters is the same).

If the two design points are the same ($Q_A = Q_D$), the plate exhibits the same in-plane and bending properties and can be considered as homogeneous with respect to the elastic properties. In this case, both points can be chosen anywhere inside the design area [2]. On the other hand, if the design points are not the same, the design area is only a restricted part of the parabola [2][8]. In this case, the plate is heterogeneous with respect to the elastic properties since in-plane and bending properties are not the same. In the following sections, three different cases will be examined: fully isotropic plates (all lamination parameters are set to zero: $Q_A = Q_D = O$), homogeneous orthotropic plates ($Q_A = Q_D$) and heterogeneous plates ($Q_A \neq Q_D$).

**Fully isotropic plates**

The problem of designing fully isotropic plates is addressed in the section. This problem has been examined by Wu et Avery [3] who found some exact solutions with 36 plies at least with standard ply orientations : 0°, ±45°, 90°. As explained in the above section, the problem is to define a laminated plate that exhibits a set of twelve given lamination parameters. The “degrees of freedom” are here the ply angles and one can *a priori* expect to find a solution with twelve plies at least. Hence, the optimization programme has first been run with required lamination parameters set to zero and with some numbers of plies lying between 7 and 18. The minimum of the objective function found in each case is reported in Fig. 2.

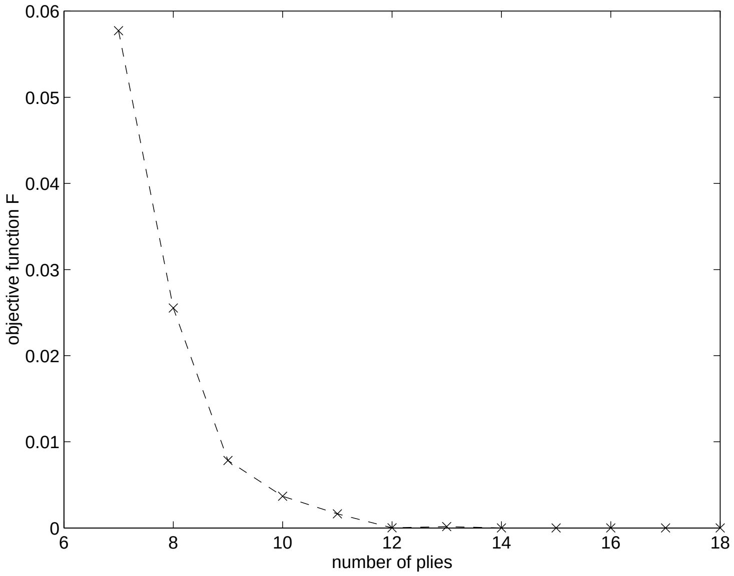



Fig. 2 : Minimum of the objective function F *vs*. n, number of plies.

As expected, the minimum of the objective function decreases as the number of plies increases and becomes very close to zero when this number is greater or equal to 12. The corresponding stacking sequence found in the particular case n=12 is reported in Table 1 (#1). The main conclusions are:

- the ply orientations are not standard and different. Note however that these angles can be approximated to their "nearest" angle multiple of 5° without strongly changing the mechanical properties [7];
- the stacking sequence is not symmetric even though the corresponding plate is uncoupled;
- since the present properties are geometric, it can be checked for any constitutive material that both $A^*$ and $D^*$ stiffness matrices exhibit isotropic properties while the $B^*$ matrix is zero;
- this result is approximated (very small residual terms instead of zeros in the $B^*$ matrix for instance) because of the numerical nature of the procedure;
- n=12 is three times lower than the exact solution shown in Ref. [3]. Moreover, other solutions are available for each value of the ply number greater than 12 [7].

**Homogeneous orthotropic plates**

The design of homogeneous orthotropic plates is addressed in this section. The case n=12 is considered. For such plates, the design points $Q_A$ and $Q_D$ are the same and can be located anywhere in the design area. Only some particular plates characterized by some properties of symmetry are presently shown. In the first example, $\overline{V_{2A}}$ and $\overline{V_{2D}}$ are set to zero. The two remaining lamination parameters are equal and set to a particular value to illustrate the approach: $\overline{V_{1A}} = \overline{V_{1D}} = 0.4$. The stacking sequence found is shown in Table 1 (#2). The conclusions are the same as in the preceding example: the laminate is not symmetric and the ply angles are different and not standard.

In order to assess the accuracy of the optimization programme, the actual lamination parameters of the stacking sequence found have been computed: $V_{1A} = 0.3993$, $V_{2A} = 0.4029$ and the ten remaining lamination parameters are less than $6 \times 10^{-3}$. As can be seen, the actual parameters are very close to the expected ones.

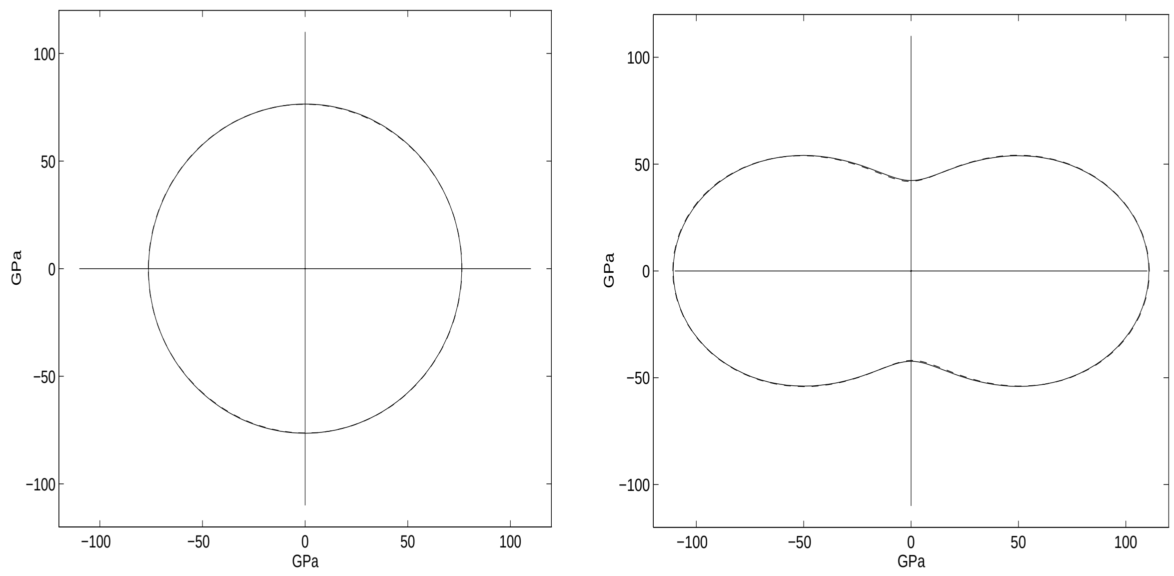
| # | $\overline{V_{1A}}$ | $\overline{V_{2A}}$ | $\overline{V_{1D}}$ | $\overline{V_{2D}}$ | In-plane property | Bending property | Stacking sequence |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | isotropy | isotropy | [0 59 107 129 50 130 178 78 179 21 69 128] |
| 2 | 0.4 | 0 | 0.4 | 0 | orthotropy | orthotropy | [13 133 62 163 169 24 28 174 111 145 46 177] |
| 3 | 0 | 0.4 | 0 | 0.4 | square symmetry | square symmetry | [96 34 167 160 87 3 86 119 16 56 98 170] |
| 4 | 0 | 0 | 0.4 | 0 | isotropy | orthotropy | [18 143 76 157 28 107 86 55 104 143 34 172] |
| 5 | 0.4 | 0 | 0 | 0 | orthotropy | isotropy | [97 159 31 35 154 152 27 187 175 174 121 53] |
| 6 | 0 | 0 | 0 | 0.4 | isotropy | square symmetry | [87 11 152 124 51 25 63 164 121 112 65 2] |
| 7 | 0 | 0.4 | 0 | 0 | square symmetry | isotropy | [32 96 162 153 84 90 5 18 94 18 76 141] |

Table 1: Examples of some particular orthotropic plates. The eight remaining lamination parameters are zero.

In the third example, only the $\overline{V_{1A}} = \overline{V_{1D}}$ parameters are set to non-zero values: $\overline{V_{1A}} = \overline{V_{1D}} = 0.4$. It can be checked that both the in-plane and bending properties of such a plate are the same and exhibit a square symmetry. The staking sequence found is given in Table 1 (#3).

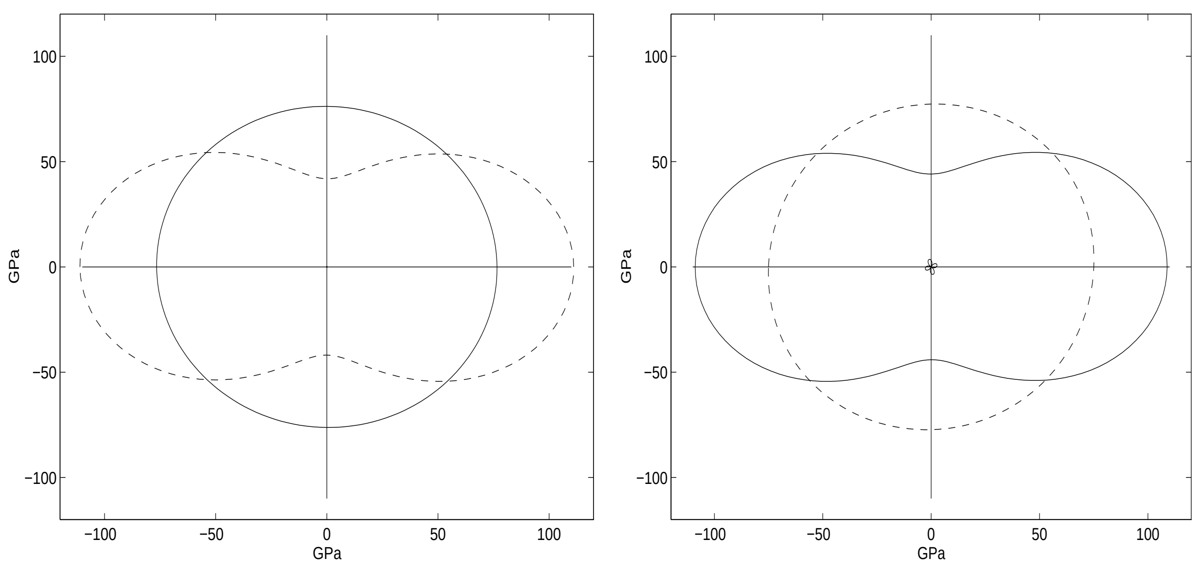
## Heterogeneous orthotropic plates

The usual so-called quasi-isotropic laminates exhibit isotropic elastic properties whereas the bending properties do not have any particular properties. It is shown through the fourth example that quasi-isotropic plates can be orthotropic with respect to the bending properties. In this example, the required lamination parameters $\overline{V_{1A}}$ and $\overline{V_{1D}}$ are set to zero to obtain in-plane isotropy whereas $\overline{V_{2D}}$ is set to a different value: $\overline{V_{2D}} = 0.4$ to obtain different in-plane and bending properties. The stacking sequence found is given in Table 1 (#4). Since $\overline{V_{2D}} = 0.4$ and $\overline{V_{2A}} = 0$, it can be checked that this fourth plate has the same bending properties as the plate in the second example.
In the fifth example (#5 in Table 1), in-plane and bending are inverted with respect to the fourth example. As a result, in-plane properties are orthotropic whereas bending properties are isotropic.
The sixth example (#6 in Table 1) is similar to the fourth one: in-plane properties are isotropic whereas bending properties exhibit a square symmetry. It can be checked that this plate has the same bending properties as the plate in the third example.




a- #1 [0 59 107 129 50 130 178 78 179 21 69 128]

b- #2 [13 133 62 163 169 24 28 174 111 145 46 177]




c- #4 [18 143 76 157 28 107 86 55 104 143 34 172]

d- #5 [97 159 31 35 154 152 27 187 175 174 121 53]

Fig. 3: Polar plot of $A^*_{11}$ (solid line), $D^*_{11}$ (dashed line) and $B^*_{11}$ (at the center) for various stacking sequences, in Gpa.

In the seventh example (#7 in Table 1), in-plane and bending properties are inverted with respect to the sixth example. As a result, in-plane properties exhibit a square symmetry whereas bending properties are isotropic.

**Polar plots on some stiffnesses**

The above results are illustrated in Fig. 3 with some polar plots of three normalized stiffnesses: $A_{11}^{*}$, $B_{11}^{*}$ and $D_{11}^{*}$. The constitutive material is a carbon/epoxy which elastic properties are taken in Ref. [1]. As can be seen, the $B_{11}^{*}$ polar plot reduces to a point in the three first examples. This clearly shows that the three corresponding plates are uncoupled. In the last example however, the $B_{11}^{*}$ plot is no more a point. This is due to the fact that the design point $Q_A$ has been chosen in the vicinity of the bound of the design area. As a result, one cannot reach exactly the desired properties [9]. In the two first cases, both $A_{11}^{*}$ and $D_{11}^{*}$ plots cannot be distinguished to the naked eyes. This illustrates the homogeneity of the plates. In the first example, these two polar plots are a circle. This is a consequence of the in-plane and bending isotropic properties of the plate. Finally, it can be seen that both in-plane and bending properties have been inverted in the two last examples.

**Plates with different in-plane and bending orthotropy axes**

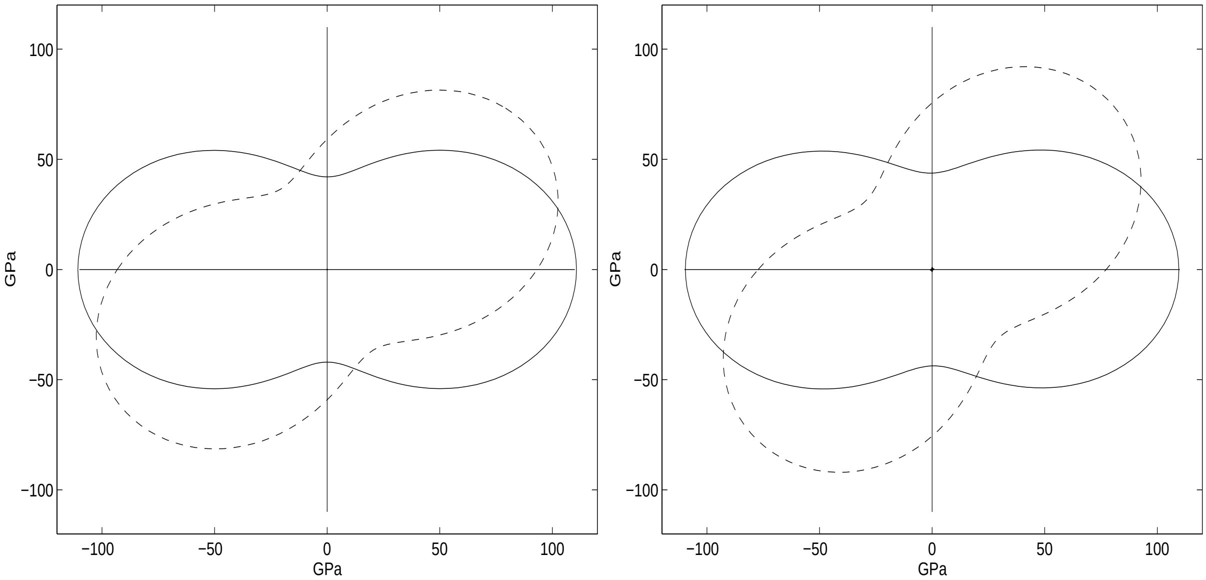
In the above examples, both in-plane and bending orthotropy axes are the same. However, one can also easily design orthotropic plates with different in-plane and bending orthotropy axes. For instance, let us consider that the orthotropy axes are the reference axes. If one wishes to design a plate similar to the second plate in Table 1, but with orthotropy axes rotated through an angle $\alpha$ between in-plane and bending properties, the only non-zero in-plane lamination parameters is $\overline{V_{1A}}$ and the required lamination parameters governing the bending properties must be chosen as follows [9]

$$\overline{V_{1D}} = \overline{V_{1A}} \cos 2\alpha, \overline{V_{3D}} = \overline{V_{1A}} \sin 2\alpha \qquad (4)$$

As an example, the programme is run with the above required parameters that have been computed in two particular cases $\alpha = 30°$ and $\alpha = 45°$. Results are shown in Table 2 and in Fig. 4.

| # | $\alpha$ | $\overline{V_{1A}}$ | $\overline{V_{1D}}$ | $\overline{V_{3D}}$ | In-plane property | Bending property | Stacking sequence |
|---|---|---|---|---|---|---|---|
| 8 | 30° | 0.4 | $0.4 \times \frac{1}{2}$ | $0.4 \times \sqrt{3}/2$ | orthotropy | orthotropy (30° rotation) | [61 9 132 16 179 151 21 151 132 178 68 24] |
| 9 | 45° | 0.4 | 0 | 0.4 | orthotropy | orthotropy (45° rotation) | [66 21 131 14 169 7 154 152 138 21 22 74] |

Table 2: Examples of two orthotropic plates with orthotropy axes rotated from each other. The nine remaining lamination parameters are zero.




a- #8 [61 9 132 16 179 151 21 151 132 178 68 24]

b- #9 [66 21 131 14 169 7 154 152 138 21 22 74]

Fig. 4: Polar plot of $A^*_{11}$ (solid line), $D^*_{11}$ (dashed line) and $B^*_{11}$ (at the center) for various stacking sequences, in Gpa.

The orthotropy axes clearly appear in both cases in Fig. 4. These two examples show that a wide freedom exists in the design of laminated plates using the present approach.

## CONCLUSION

The main features of a procedure allowing the design of thin orthotropic plates verifying required elastic properties are described in this paper and several examples illustrate the relevance of the approach. In general term, the idea is to built up an objective function with the twelve lamination parameters describing the required properties and to minimize this function with respect to the ply angles that are considered as continuous. The main conclusions of the paper are:

- the particular case n=12 plies has been studied because twelve constraints have been considered: the twelve required lamination parameters. A solution has been found for each of the examples. In a more general study [9], it is shown that no simple general rule can be obtained concerning the minimum number of plies of a laminated plate verifying required elastic properties since this number depends on the value of the twelve required parameters;
- the solutions found are not symmetric even though the plates are uncoupled. This illustrates the fact that the condition of symmetry is sufficient and not necessary even though most of the usual laminated plates manufactured are symmetric;
- the particular properties highlighted in this paper are geometric and independent of the constitutive material of the plies;
- the ply orientations are not standard and different from one ply to another. This is probably the main drawback of the present approach but if seems that it is the price one has to pay to obtain required elastic properties with such a low number of plies;

- it seems that most of the particular properties of the thin laminated plates shown in this paper have not been obtained with usual standard design rules until now.

Some of the above properties could be considered as academic only. However the control of the stiffness properties of composite structures is of prime importance in many optimization problems and the above examples show that this optimal properties can be easily controlled with the procedure presented in this paper.

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